STAT 315 Midterm Sheet

Shaquille Que

Spring 2019

§1 Probability

Definition 1.1. Sets $A_1, \ldots, A_k$ are **exhaustive** if $A_1 \cup \cdots \cup A_k = S$

Definition 1.2. **Probability** is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. if $A_1, A_2, \ldots$ are events and $A_i \cap A_j = \emptyset, i \neq j$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

for each positive integer k and

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

for an infinite, but countable, number of events.

Definition 1.3. The **conditional probability** of an event A, given that event B has occurred, is defined by $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$.

Definition 1.4. Events A and B are **independent** if and only if $P(A \cap B) = P(A)P(B)$.

Theorem 1.5 (Baye's Theorem)
Let $B_1, \ldots, B_m$ be a partition of S.
Then $P(B_k \cap A) = \dfrac{P(A \mid B_k)P(B_k)}{\sum_{i=1}^{m} P(A \mid B_i)P(B_i)}$.

§2 Random Experiments

Definition 2.1. Given a **random experiment** with an outcome space S, a function X that assigns one and only one real number $X(s) = x$ to each element s in S is called a random variable. The space of X is the set of real numbers $\{x \mid X(s) = x, s \in S\}$, where $s \in S$ means that the element s belongs to the set S.

Definition 2.2. A random variable of the **discrete type** has a countable number of values in S

Definition 2.3. The **probability mass function** (probability density function, frequency function, probability function) $f(x)$ is the probability $P(X = x)$ and satisfies

1. $f(x) > 0$ for $x \in S$
2. $\sum_{x \in S} f(x) = 1$
3. $P(X \in A) = \sum_{x \in A} f(x)$, where $A \subset S$.

Definition 2.4. The **support** S of x contains all the outcomes with positive probabilities associated with X.

Definition 2.5. The **cumulative distribution function** of X is given by $F(x) \stackrel{\text{def}}{=} P(X \le x)$.

§3 Distributions

Definition 3.1. **Bernoulli distribution** has $S_x = \{0, 1\}$, denoted as $X \sim \text{Bern}(p)$, and satisfies $P(X = 1) = p$, $f(x) = p^x(1 - p)^{1-x}$. $\mu = p$ and $\sigma^2 = pq$.

Definition 3.2. **Binomial distribution** denoted as $X \sim \text{Bin}(n, p)$ with $f(x) = \binom{n}{x} p^x(1 - p)^{n-x}$

Definition 3.3. **Hypergeometric distribution** denoted $X \sim \text{Hyper}(N_1, N_2, n)$ with $f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$.

Definition 3.4. **Geometric distribution**. Try until first success. $f(x) = q^{x-1}p$ with $\mu = \frac{1}{p}$ and $\sigma^2 = \frac{q}{p}$

Definition 3.5. **Poisson distribution**. $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$, where $\lambda > 0$. $\mu = \sigma^2 = \lambda$

Definition 3.6. An **approximate Poisson process** with parameter $\lambda > 0$ satisfies

1. The numbers of occurrences in non-overlapping subintervals are independent.
2. The probability of exactly one occurrence in a sufficiently short subinterval of length h is λh.
3. The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Definition 3.7. **Exponential distribution**. $f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $0 \le x < \infty$. $F(w) = 1 - e^{-\lambda w}$. $M(t) = \frac{1}{1-\theta t}$, $M'(t) = \frac{\theta}{(1-\theta t)^2}$, $M''(t) = \frac{2\theta^2}{(1-\theta t)^3}$

§4 Expected Values

Definition 4.1. The **expected value** of $u(X)$ for the *discrete* random variable X is given by

$$\mathbb{E}[u(X)] = \sum_{x \in S} u(X)f(x)$$

if the sum exists, i.e., $\mathbb{E}[u(X)]$ is a weighted mean of $u(X)$.

Definition 4.2. **Mean**. $\mu = \mathbb{E}[X]$

Definition 4.3. **Variance**. $\sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ by linearity of expectation.

Proposition 4.4

Let X be a random variable with mean μ_X and variance σ_X^2. Then random variable $Y = aX + b$ has mean $\mu_Y = a\mu_X + b$ and variance $\sigma_Y^2 = a^2\sigma_X^2$ for constants a, b.

§5 Moment Generating Function

Definition 5.1. The **rth moment** of a distribution about b is $\mathbb{E}[(X-b)^r] = \sum_{x \in S}(x - b)^r f(x)$

Definition 5.2. The **moment generating function** of a discrete random variable X is $M(t) = \mathbb{E}(e^{tX}) = \sum_{x \in S} e^{tX} f(x)$ if the value exists and is finite for $-h < t < h$.

Proposition 5.3

We have the following:

- $M^{(r)}(t) = \sum_{x \in S} x^r e^{tx} f(x)$
- $M'(0) = \sum_{x \in S} x f(x) = \mathbb{E}(X)$
- $M''(0) = \sum_{x \in S} x^2 f(x) = \mathbb{E}(X^2)$

§6 Continuous Random Variables

Let X be a continuous random variable defined on the interval $[a, b]$.

Definition 6.1. The **probability density function** of X is an integrable function $f(x)$ satisfying

1. $f(x) \geq 0$ for $x \in S$
2. $\displaystyle\int_S f(x)dx = 1$
3. if $(a, b) \in S$, the $P(a < X < b) = \displaystyle\int_a^b f(x)dx$

Definition 6.2. The **cumulative distribution function** is given by $F(x) = \displaystyle\int_{-\infty}^{x} f(t)dt$ for $-\infty < x < \infty$.

Definition 6.3. The **expected value** of $u(X)$ is $\mathbb{E}[u(x)] = \displaystyle\int_{-\infty}^{\infty} u(x)f(x)dx$.

Definition 6.4. The **moment generating function** of X is $\displaystyle\int_{-\infty}^{\infty} e^{tx}f(x)dx$, $-h < t < h$.

Proposition 6.5

Let random variables X, Y be such that $Y = aX + b$ for constants a, b. Then

$$M_y(t) = e^{tb}M_x(ta).$$

§7 Multivariate

Proposition 7.1

Let $X_1, X_2, \ldots, X_n$ be n independent discrete random variables with individual probability mass functions $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$. The joint probability mass function

$$f(X_1, X_2, \ldots, X_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$$

Definition 7.2. **Independent and identically distributed**.

1. $X_1, \ldots, X_n$ are independent
2. All $X_1, \ldots, X_n$ have the same distribution