MATH 331 Lecture Notes

Taught by Dr. Stephen Semmes

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These are my notes for Rice University's MATH 331: Honors Analysis, taught by Dr. Stephen Semmes. This file was created in LaTeXand uses Evan Chen's `evan.sty` package. Any mistake herein is my own. Please let me know of any errors by emailing me at stq1@rice.edu.

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§1 August 26, 2019

§1.1 Metric Space

Definition

A **metric space** is a set $\mathcal{M}$ together with a nonnegative real-valued function $d(x, y)$ defined for $x, y \in \mathcal{M}$ that satisfies the following properties:

1. $d(x, y) = 0$ if and only if $x = y$;
2. **(symmetry)** $d(x, y) = d(y, x)$ for every $x, y \in \mathcal{M}$;
3. **(triangle inequality)** $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in \mathcal{M}$.

§1.1.1 Examples

Each of the following can be easily proven to be a metric space.

1. The standard Euclidean metric defined on the real line $\mathbb{R}$ by $d(x, y) = |x - y|$ for every $x, y \in \mathbb{R}$.
2. The discrete metric defined on any set with $d(x, y) = 0$ when $x = y$ and $d(x, y) = 1$ when $x \neq y$.
3. The standard Euclidean metric defined on $\mathbb{R}^n$ by $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$[1].
4. The taxicab metric defined on $\mathbb{R}^n$ by $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.
5. If $(\mathcal{M}, d(x, y))$ is a metric space and if Y is any subset of $\mathcal{M}$, then the restriction of $d(x, y)$ to $x, y \in Y$ defined a metric on Y.

§1.2 Suprema

§1.2.1 Definition

Let A be a set of real numbers. A real number b is said to be an **upper bound** for A if for every $a \in A$, $a \leq b$. A real number c is said to be the **least upper bound** or **supremum** of A if c is an upper bound of A and if for every upper bound $b \in \mathbb{R}$ of A, we have $c \leq b$. It is easy to see that the supremum of A is unique when it exists, in which case it may be denoted as $\sup A$.

Proposition 1.1 (Least Upper Bound Property)

The least upper bound property for the real line states that if A is a nonempty subset of $\mathbb{R}$ with an upper bound in $\mathbb{R}$, then A has a supremum.

[1]This is well-known to be a metric on $\mathbb{R}^n$. See Rudin's book.

§1.3 Infima

§1.3.1 Definition

Let A be a set of real numbers. A real number b is said to be a **lower bound** for A if for every $a \in A$, $a \geq b$. A real number c is said to be the **greatest lower bound** or **infimum** of A if c is a lower bound of A and if for every lower bound $b \in \mathbb{R}$ of A, we have $c \geq b$. If $A \in \mathbb{R}$ is a nonempty set with a lower bound in $\mathbb{R}$, then the infimum of A exists. More precisely, one can show that the infimum of A is given by the supremum of the set of lower bounds for A, as in Rudin's book. Alternately, if $-A = \{-a \mid a \in A\}$, then one can check that $\inf A = -\sup(-A)$.

§1.4 Properties of $\mathbb{R}$

1. **Archimedean property on** $\mathbb{R}$. If $x, y \in \mathbb{R}$ and $x, y > 0$, then there is a positive integer n such that $y < nx$.

2. If $a, b \in \mathbb{R}$, and $a < b$, then there is a rational number r such that $a < r < b$.

Theorem 1.2

If $x \in \mathbb{R}$, $x > 0$ and n is a positive integer, then there is a unique $y \in \mathbb{R}$ with $y > 0$ and $y^n = x$.

Proof. It is easy to verify directly that y is unique when it exists. To get existence, consider the set E of nonnegative numbers t such that $t^n < x$. Note that $0 \in E$, so that $E \neq \emptyset$. One can check that E has an upper bound in $\mathbb{R}$. Thus, the supremum of E exists in $\mathbb{R}$. One can show that $y = \sup E$ has the desired properties. □

§2 August 27, 2019

§2.1 Openness

Definition 2.1. The **open ball** in $\mathcal{M}$ centered at a point $x \in \mathcal{M}$ with radius $r > 0 \in \mathbb{R}$ is defined by

$$B(x, r) = \{y \in \mathcal{M} \mid d(x, y) < r\}.^{2}$$

Definition 2.2. A subset U of $\mathcal{M}$ is said to be an **open set** in $\mathcal{M}$ (with respect to d) if for every $x \in U$, there is a positive real number r such that $B(x, r) \subseteq U$.

Corollary 2.3

The empty set $\emptyset$ and $\mathcal{M}$ itself are automatically open subsets of $\mathcal{M}$.

Proposition 2.4

If $p \in \mathcal{M}$ and t is a positive real number, then $B(p, t)$ is an open set in $\mathcal{M}$.

[2] Other authors refer to the ball as a *neighborhood*.

Proof. Let $p \in \mathcal{M}$ and $t > 0$ be given. Also let x be an arbitrary element of $B(p,t)$. Put $r = t - d(p,x)$ and note that $r > 0$ because $x \in B(p,t)$. We will show that $B(x,r) \subseteq B(p,t)$.

To show this, let y be an arbitrary element of $B(x,r)$. Using the triangle inequality, we get that $d(p,y) \leq d(p,x) + d(x,y) < d(p,x) + r$ since $y \in B(x,r)$. Since $d(p,x) + r = d(p,x) + t - d(p,x) = t$, we have $d(p,y) < t$ so that $y \in B(p,t)$ as desired. □

Proposition 2.5

If $p \in \mathcal{M}$ and t is a nonnegative real number, then $V(p,t) = \{q \in \mathcal{M} \mid d(p,q) > t\}$ is an open subset of $\mathcal{M}$.

Sketch of proof. Let $x \in U(p,t)$ be given and put $r = d(p,x) - t > 0$. One can check that $B(x,r) \subseteq V(p,t)$ using the triangle inequality. □

§2.1.1 Limit Points

Definition 2.6. A point $p \in \mathcal{M}$ is said to be a **limit point** of a subset E of $\mathcal{M}$ if for every $r > 0$, there is a point $q \in E$ such that $d(p,q) < r$ and $p \neq q$.

Proposition 2.7

If $p \in \mathcal{M}$ is a limit point of a subset E of $\mathcal{M}$, then for each $r > 0 \in \mathbb{R}$, $E \cap B(p,r)$ has infinitely many elements.

Proof. Let $r > 0$ be given. Suppose, for the sake of contradiction, that $E \cap B(p,r)$ has only finitely many elements. Let $x_1, \ldots, x_n$ be a list of elements of $E \cap B(p,r)$, not including p, where $n \in \mathbb{Z}^+$.

Let $t = \min\limits_{1 \leq i \leq n} d(p,x_i)$ and note that $t > 0$ because $d(p,x_i) > 0$ for each $i = 1, \ldots, n$ by construction. Under these conditions, $E \cap B(p,t)$ is either empty or contains only p. This contradicts the hypothesis that p is a limit point of E. □

Corollary 2.8

If $E \subseteq \mathcal{M}$ has only finitely many elements, then E has no limit points.

§2.2 Closedness

Definition 2.9. A susbet E of $\mathcal{M}$ is said to be a **closed set** of $\mathcal{M}$ with respect to d if E contains all of its limit points in $\mathcal{M}$. More precisely, this means that if $p \in \mathcal{M}$ is a limit point of E in $\mathcal{M}$, then $p \in E$.

Corollary 2.10

The empty set $\emptyset$ and $\mathcal{M}$ itself are automatically closed sets in $\mathcal{M}$.

Proposition 2.11

A subset U of $\mathcal{M}$ is an open set if and only if $M \setminus U = \{x \in \mathcal{M} \mid x \notin U\}$ is a closed set in $\mathcal{M}$.

Proof. Suppose that $U \subseteq \mathcal{M}$ is an open set. We will show that $M \setminus U$ is a closed set in $\mathcal{M}$. Let arbitrary $x \in U$. Then there exists $r > 0$ such that $B(x, r) \subseteq U$ because U is an open set. Equivalently, this means that $B(x, r) \cap (M \setminus U) = \emptyset$. This implies that x is not a limit point of $M \setminus U$ in M. It follows that every limit point of $M \setminus U$ in M is an element of $M \setminus U$, as desired.

Now suppose that $M \setminus U$ is a closed set in M. We will show that U is an open set in M. Let arbitrary $x \in U$. Note that x is not a limit point of $M \setminus U$ in M since otherwise, x would be an element of $M \setminus U$ because $M \setminus U$ is a closed set in M. This implies that there is an $r > 0$ such that $B(x, r) \cap (M \setminus U)$ is either empty or contains only x. However, because $x \in U$, we get that $B(x, r) \cap (M \setminus U) = \emptyset$. This implies that $B(x, r) \subseteq U$, as desired. □

Corollary 2.12 (Complementariness of Open and Closed Sets)

A subset E of M is a closed set if and only if $M \setminus E$ is an open set.

§3 August 30, 2019

Proposition 3.1 (Intersection of Finitely Many Open Sets Is Open)

If $U_1, \dots, U_n$ are finitely many open subsets of $\mathcal{M}$, then their intersection $\bigcap_{i=1}^{n} U_i$ is an open set in $\mathcal{M}$.

Proof. Let x be an arbitrary element of $\bigcap_{i=1}^{n} U_i$. Thus, for each $i = 1, \dots, n$, we have that $x \in U_i$. So for each $i = 1, \dots, n$, ther is an $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ because U_i is an open set in $\mathcal{M}$.

Let $r = \min(r_1, \dots, r_n) > 0$. Note that for each $i = 1, \dots, n$, $B(x, r) \subseteq B(x, r_i) \subseteq U_i$, which implies that $B(x, r) \subseteq \bigcap_{i=1}^{n} U_i$ as desired. □

Corollary 3.2 (Union of Finitely Many Closed Sets Is Closed)

If $E_1, \dots, E_n$ are finitely many closed subsets of $\mathcal{M}$, then their union $\bigcup_{i=1}^{n} E_i$ is a closed set in $\mathcal{M}$.

Proof. Let $U_i = M \setminus E_i$ for each $i = 1, \dots, n$. Note that U_i is an open set in $\mathcal{M}$ for every $i = 1, \dots, n$ by Proposition 2.11. Hence, $\bigcap_{i=1}^{n} U_i$ is an open set by Proposition 3.1.

Observe that $\mathcal{M} \setminus \left(\bigcap_{i=1}^{n} U_i \right) = \bigcup_{i=1}^{n} (\mathcal{M} \setminus U_i) = \bigcup_{i=1}^{n} E_i$.

It follows that $\bigcup_{i=1}^{n} E_i$ is a closed set in $\mathcal{M}$ by Proposition 2.11. □

Proposition 3.3

Let A be a nonempty set, and suppose that for each $\alpha \in A$, U_α is an open set in $\mathcal{M}$. Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set in $\mathcal{M}$.

Proof. Let x be an arbitrary element of $\bigcup_{\alpha \in A} U_\alpha$. By definition, there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. Because U_{α_0} is an open set in $\mathcal{M}$, there is an $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$. This implies that $B(x, r) \subseteq \bigcup_{\alpha \in A} U_\alpha$, as desired. □

Corollary 3.4

Let A be a nonempty set, and suppose that E_α is a closed set in $\mathcal{M}$ for each $\alpha \in A$. Then $\bigcap_{\alpha \in A} E_\alpha$ is a closed set in $\mathcal{M}$.

Proof. For each $\alpha \in A$, $U_\alpha = \mathcal{M} \setminus E_\alpha$ is an open set in $\mathcal{M}$. Thus, $\bigcup_{\alpha \in A} U_\alpha$ is an open set in $\mathcal{M}$ by Proposition 3.3. Observe that $\mathcal{M} \setminus \left(\bigcup_{\alpha \in A} U_\alpha \right) = \bigcap_{\alpha \in A} (\mathcal{M} \setminus U_\alpha) = \bigcap_{\alpha \in A} E_\alpha$.

It follows that $\bigcap_{\alpha \in A} E_\alpha$ is a closed set in $\mathcal{M}$ by Proposition 2.11. □

Direct proof. Suppose that $x \in \mathcal{M}$ is a limit point of $\bigcap_{\alpha \in A} E_\alpha$. If $\beta \in A$, then $\bigcap_{\alpha \in A} E_\alpha \subseteq E_\beta$. It is easy to see that x is a limit point of E_β.

This implies that $x \in E_\beta$ because E_β is a closed set. Thus, $x \in \bigcap_{\beta \in A} E_\beta = \bigcap_{\alpha \in A} E_\alpha$, as desired. □

§3.1 Closure

Definition 3.5. The **closure** of a subset E of $\mathcal{M}$ is the set $\bar{E}$ of $x \in \mathcal{M}$ such that $x \in E$ or x is a limit point of E in $\mathcal{M}$, or both.

Note that $E \subseteq \bar{E}$ automatically.

If E is a closed set, then it is easy to see that $\bar{E} = E$. Conversely, if $E = \bar{E}$, then E contains all of its limit points in $\mathcal{M}$, so that E is a closed set in $\mathcal{M}$.

Proposition 3.6 (Closures Are Closed)

If E is any subset of $\mathcal{M}$, then $\bar{E}$ is a closed set in $\mathcal{M}$.

Proof. It suffices to show that $\mathcal{M} \setminus \bar{E}$ is an open set in $\mathcal{M}$. Let $x \in \mathcal{M} \setminus \bar{E}$. Then $x \notin E$ and x is not a limit point of E. The latter implies that there is an $r > 0$ such that $B(x, r) \cap E = \emptyset$ or $\{x\}$. Because $x \notin E$, we have that $B(x, r) \cap E = \emptyset$.

We can check that $B(x, r) \cap \bar{E} = \emptyset$. This would imply that $B(x, r) \subseteq M \setminus \bar{E}$, as desired. ☐

§4 September 4, 2019

Proposition 4.1 (Supremum and Infimum in Closure)

If $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and A has an upper bound in $\mathbb{R}$, then $\sup A \in \bar{A}$. Similarly, if $B \subseteq \mathbb{R}$, $B \neq \emptyset$, and B has a lower bound in $\mathbb{R}$, then $\inf B \in \bar{B}$.

Proof. Let $y = \sup A$. If $y \in A$, then $y \in \bar{A}$. Suppose $y \notin A$. Then for every $r > 0$, there is an $x \in A$ such that $y - r < x < y$, for otherwise, $y - r$ would be an upper bound of A. This implies that y is a limit point of A, so $y \in \bar{A}$. The other case is proven similarly. ☐

§4.1 Compactness

Definition 4.2. Let a and b be real numbers with $a < b$. Then
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
We may also use the notation $[a, b]$ when $a = b$. Note that (a, b) is an open set in $\mathbb{R}$ and that $[a, b]$ is a closed set in $\mathbb{R}$ with respect to the standard Euclidean metric on $\mathbb{R}$.

Definition 4.3. An **open covering** of a subset K of $\mathcal{M}$ in $\mathcal{M}$ is a family $\{U_\alpha\}_{\alpha \in A}$ of open subsets of $\mathcal{M}$ such that $K \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Definition 4.4. A subset K of $\mathcal{M}$ is said to be **compact** in $\mathcal{M}$ if every open covering of K in $\mathcal{M}$ can be reduced to a finite subcovering. More precisely, this means that for every open covering $\{U_\alpha\}_{\alpha \in A}$ of K in $\mathcal{M}$, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Remark 4.5. If K has only finitely many elements, K is trivially compact.

Example 4.6

For $\mathcal{M} = \mathbb{R}$ with the standard Euclidean metric,

1. $K = \{0\} \cup \{\frac{1}{i} \mid i \in \mathbb{Z}^+\}$ is compact in $\mathbb{R}$.
2. $(0, 1)$ is not compact in $\mathbb{R}$.

Proofs. We give the proofs for the examples.

1. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of K in $\mathbb{R}$. Note that $0 \in K \subseteq \bigcup_{\alpha \in A} U_\alpha$, so that there is an $\alpha_0 \in A$ such that $0 \in U_{\alpha_0}$. Because U_{α_0} is an open set in $\mathbb{R}$,

there is an $r > 0$ such that $B(0, r) \subseteq U_{\alpha_0}$. In this case, $B(0, r) = (-r, r)$. Observe that $\frac{1}{i} \in (-r, r)$ when $i > \frac{1}{r}$. It follows that $(-r, r)$ contains all but finitely many elements of K. One can use this to find a finite subcovering of K from $\{U_\alpha\}_{\alpha \in A}$. □

2. Let $U_i = (\frac{1}{i}, 1)$ for $i \in \mathbb{Z}^+$. It is easy to see that $\{U_i\}_{i\in\mathbb{Z}^+}$ is an open covering of $(0, 1)$ in $\mathbb{R}$. More precisely, U_i is an open set in $\mathbb{R}$ for each $i \in \mathbb{Z}^+$ as before. We also have that $\bigcup_{i\in\mathbb{Z}^+} U_i = \bigcup_{i=1}^{\infty} (\frac{1}{i}, 1) = (0, 1)$. However, one can check that there is no finite subcovering of $(0, 1)$ from this open covering. □

Definition 4.7. A subset E of $\mathcal{M}$ is **bounded** if there is a point $p \in \mathcal{M}$ and a positive real number r such that $E \subseteq B(p, r)$. If $\mathcal{M} = \emptyset$, we consider $\emptyset$ to be a bounded subset of $\mathcal{M}$.

Proposition 4.8 (Compactness Implies Upper Bound)
If $K \subseteq \mathcal{M}$ is compact and $p \in \mathcal{M}$, then there is an $r > 0$ such that $K \subseteq B(p, r)$.

Proof. Consider the family of open balls $B(p, i)$ with $i \in \mathbb{Z}$. Note that $B(p, i)$ is an open set in $\mathcal{M}$ for every $i \in \mathbb{Z}^+$. It is easy to see that $K \subseteq \bigcup_{i=1}^{\infty} B(p, i)$. Thus, $\{B(p, i)\}_{i\in\mathbb{Z}^+}$ is an open covering of K in $\mathcal{M}$. Since K is compact, there is a finite subcovering of K from this open covering. This means that there are finitely many positive integers $i_1, \ldots, i_n$ such that $K \subseteq \bigcup_{\ell=1}^{n} B(p, i_\ell)$. Taking $r = \max(i_1, \ldots, i_n) \in \mathbb{Z}^+$, it is easy to see that $\bigcup_{\ell=1}^{n} B(p, i_\ell) = B(p, r)$ so that $K \subseteq B(p, r)$, as desired. □

Definition 4.9. A subset E of $\mathcal{M}$ is said to have the **limit point property** if for every subset L of E such that L has infinitely many elements, there is a point $p \in E$ such that P is a limit point of L in $\mathcal{M}$.

Remark 4.10. If $E \subseteq \mathcal{M}$ has only finitely many elements, then E trivially satisfies the limit point property.

§5 September 6, 2019

Proposition 5.1 (Compactness Implies Limit Point Property)
If $K \subseteq \mathcal{M}$ is compact, then K has the limit point property.

Proof. Let L be an arbitrary infinite subset of K. Suppose, on the contrary, that for every $p \in K$, p is not a limit point of L in $\mathcal{M}$. This means that for every $p \in K$, there is a an $r_p > 0$ such that $B(p, r_p) \cap L \subseteq \{p\}$. Note that $K \subseteq \bigcup_{p\in K} B(p, r_p)$ because $p \in B(p, r_p)$ for every $p \in K$. It follows that the collection of open balls of the form $B(p, r_p)$ with $p \in K$ is an open covering of K in $\mathcal{M}$. Since K is compact, there are finitely many $p_1, \ldots, p_n \in K$

such that $K \subseteq \bigcup_{i=1}^{n} B(p_i, r_{p_i})$. In particular, $L \subseteq \bigcup_{i=1}^{n} B(p_i, r_{p_i})$. By construction, L has at most one element in $B(p_i, r_{p_i})$, so L must have at most n elements. This contradicts the assumption that L is infinite. □

Proposition 5.2 (Limit Point Property Implies Upper Bound)

If $E \subseteq \mathcal{M}$ has the limit point property and if $p \in \mathcal{M}$, then there is an $r > 0$ such that $E \subseteq B(p, r)$.

Proof. Let $p \in \mathcal{M}$. Suppose, on the contrary, that for each $r > 0$, we have $E \not\subseteq B(p, r)$. Then for each $i \in \mathbb{Z}^+$, there is an element $x_i \in E$ such that $d(p, x) > i$. Let $L = \{x_i \mid i \in \mathbb{Z}^+\}$.

We show that L is infinite. Otherwise, if L had finitely many elements, then this implies that there is an $x \in E$ such that $x = x_i$ for infinitely many $i \in \mathbb{Z}^+$. This would imply that $i < d(p, x)$ for infinitely many $i \in \mathbb{Z}^+$, which is a contradiction. Thus, L is infinite.

Since E has the limit point property, there is an element q of E such that q is a limit point of L in $\mathcal{M}$. By Proposition 2.7, there are infinitely many elements of L in $B(q, 1)$. It follows that $x_i \in B(q, 1)$ for infinitely many $i \in \mathbb{Z}^+$. Note that $d(p, x_i) \le d(p, q) + d(q, x_i)$ for every $i \in \mathbb{Z}^+$ by the triangle inequality. It follows that $d(p, x_i) < d(p, q) + 1$ for infinitely many $i \in \mathbb{Z}^+$, and so $i < d(p, q) + 1$ for infinitely many $i \in \mathbb{Z}^+$. This is a contradiction since the RHS is a fixed value. □

Proposition 5.3 (Compactness Implies Closedness)

If $K \subseteq \mathcal{M}$ is compact, then K is a closed set in $\mathcal{M}$.

Proof. Let $p \in \mathcal{M}$ be a limit point of K in $\mathcal{M}$. Suppose, on the contrary, that $p \notin K$. Note that for each $r > 0$, $V(p, r) = \{x \mid x \in \mathcal{M}, d(p, x) > r\}$ is an open set in $\mathcal{M}$. Observe that $\bigcup_{i=1}^{\infty} V(p, \frac{1}{i}) = \mathcal{M} \setminus \{p\} \supseteq K$. Thus, $\{V(p, \frac{1}{n})\}_{i \in \mathbb{Z}^+}$ is an open covering of K in $\mathcal{M}$. Since K is compact, there must be finitely many positive integers $i_1, \ldots, i_n$ such that $K \subseteq \bigcup_{\ell=1}^{n} V(p, \frac{1}{i_\ell})$. Let $J = \max(i_i, \ldots, i_n)$. Then $V(p, \frac{1}{i_\ell}) \subseteq V(p, \frac{1}{J})$ for each $\ell = 1, \ldots, n$, so that $\bigcup_{\ell=1}^{n} V(p, \frac{1}{i_\ell}) = V(p, \frac{1}{J})$. Thus, $K \subseteq V(p, \frac{1}{J})$, which contradicts the assumption that p is a limit point in K. □

§6 September 9, 2019

Proposition 6.1

If $K \subseteq \mathcal{M}$ is compact and $E \subseteq \mathcal{M}$ is closed with $E \subseteq K$, then E is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of E in $\mathcal{M}$. Note that $M \setminus E$ is an open set in $\mathcal{M}$ by Proposition 2.11. Since $\left(\bigcup_{\alpha \in A}\right) \cup (\mathcal{M} \setminus E) \supseteq E \cup (\mathcal{M} \setminus E) = \mathcal{M} \supseteq K$.

It follows that $\left(\bigcup_{\alpha\in A}\right) \cup (\mathcal{M} \setminus E)$ is an open covering of K in $\mathcal{M}$. Since K is compact, this open covering can be reduced to a finite subcovering, i.e. there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \cup (M \setminus E)$. Since $E \subseteq K$, it follows that $E \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, as desired. □

Definition 6.2. Let $Y \subseteq \mathcal{M}$. A subset $E \subseteq Y$ is **relatively open** to Y if E is an open set in Y, where Y is considered as a metric space itself.

More precisely, let $x \in Y$, $r > 0$, and $B_Y(x,r) = \{y \mid y \in Y, d(x,y) < r\} = B(x,r) \cap Y$. Then $E \subseteq Y$ is relatively open in Y if and only if for every $x \in E$, there is an $r > 0$ such that $B_Y(x,r) \subseteq E$.

Proposition 6.3 (Subsets of Compact Sets Are Closed)
Let $E \subseteq Y \subseteq \mathcal{M}$. Then E is relatively open in Y if and only if there is an open set U in $\mathcal{M}$ such that $E = U \cap Y$.

Proof. Let U be an open subset of $\mathcal{M}$. We shall prove that $E = U \cap Y$ is a relatively open open subset of Y. Let x be an arbitrary element of E. Note that $x \in U$. Since U is open, there is an $r > 0$ such that $B(x,r) \subseteq U$. This implies that $B_Y(x,r) = B(x,r) \cap Y \subseteq U \cap Y = E$, as desired.

For the reverse direction, suppose that $E \subseteq Y$ is relatively open in Y. Then for every $x \in E$, there is an $r_x > 0$ such that $B_Y(x,r_x) \subseteq E$. Let $U = \bigcup_{x\in E} B(x,r_x)$. Observe that U is an open set in $\mathcal{M}$ by Proposition 3.3. We will show that $E = U \cap Y$. Let $x \in E$. Then $x \in B(x,r_x) \subseteq U$ and $x \in E \subseteq Y$, so that $x \in U \cap Y$. This shows that $E \subseteq U \cap Y$. To show that $U \cap Y \subseteq E$, observe that $U \cap Y = \left(\bigcup_{x\in E} B(x,r_x)\right) \cap Y = \bigcup_{x\in E} (B(x,r_x) \cap Y) = \bigcup_{x\in E} B_Y(x,r_x)$. It follows that $U \cap Y \subseteq E$ because $B_Y(x,r_x) \subseteq E/$. This concludes the proof. □

§7 September 11, 2019

Proposition 7.1 (Transitivity of Compactness)
$K \subseteq Y$ is compact as a subset of Y if and only if K is compact as a subset of $\mathcal{M}$.

Proof. Suppose that K is compact as a subset of $\mathcal{M}$. We will show that K is compact as a subset of Y. Let $\{E_\alpha\}_{\alpha\in A}$ be an arbitrary open covering of K in Y so that E_α is a relatively open set in Y for each $\alpha \in A$, and $K \subseteq \bigcup_{\alpha\in A} E_\alpha$. For each $\alpha \in A$, there is an open subset U_α of $\mathcal{M}$ such that $E_\alpha = U_\alpha \cap Y$ by Proposition 6.3. Note that $K \subseteq \bigcup_{\alpha\in A} E_\alpha \subseteq \bigcup_{\alpha\in A} U_\alpha$ because $E_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Thus, $\{U_\alpha\}_{\alpha\in A}$ is an open covering of K in $\mathcal{M}$. Since K is compact in $\mathcal{M}$, there are finitely many indices

$\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Remember that $K \subseteq Y$ by hypothesis. Hence, $K \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \cap Y = \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} E_{\alpha_i}$ as desired.

Now suppose that K is compact as a subset of Y. We will show that K is also compact as a subset of $\mathcal{M}$. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of K in $\mathcal{M}$. For each $\alpha \in A$, let $E_\alpha = U_\alpha \cap Y$ and note that E_α is relatively open in Y because U_α is an open set in $\mathcal{M}$, and by Proposition 6.3. Also, $K \subseteq \bigcup_{\alpha \in A} U_\alpha$ by assumption. Since $K \subseteq Y$ by assumption also, $K \subseteq \left(\bigcup_{\alpha \in A} U_\alpha\right) \cap Y = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \bigcup_{\alpha \in A} E_\alpha$. Thus, $\{E_\alpha\}_{\alpha \in A}$ is an open covering of K in Y. Since K is compact in Y, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{i=1}^{n} E_{\alpha_i}$. This implies that $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ because $E_{\alpha_i} \subseteq U_i$ by definition of E_{α_i}, as desired. □

§8 September 13, 2019

Proposition 8.1 (Intersection of Infinitely Nested Closed Intervals Is Nonempty)
If $I_1, I_2, I_3, \ldots$ is an infinite sequence of closed intervals in the real line such that $I_{i+1} \subseteq I_i$ for each $i \in \mathbb{Z}^+$, then $\cap_{i=1}^{\infty} I_i \neq \emptyset$.

Proof. For each $i \in \mathbb{Z}^+$, there are real numbers a_i and b_i such that $a_i \leq b_i$ and $I_i = [a_i, b_i]$ by definition of a closed interval in $\mathbb{R}$. We also have that $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$ for each $i \in \mathbb{Z}^+$ because $I_{i+1} \subseteq I_i$.

We claim that for every $i, j \in \mathbb{Z}^+$, we have $a_i \leq b_j$. We consider two cases. If $i \leq j$, then $a_i \leq a_j \leq b_j$. Otherwise, if $i \geq j$, then $a_i \leq b_i \leq b_j$. This proves the claim.

Let A be the set of a_i for $i \in \mathbb{Z}^+$. Note that the supremum of A exists in $\mathbb{R}$ because $A \neq \emptyset$ by construction, and A has an upper bound in $\mathbb{R}$ by the lemma. We also have that $a_i \leq \sup A$ for each $i \in \mathbb{Z}^+$ because $a_i \in A$. We also have that $\sup A \leq b_j$ for each $j \in \mathbb{Z}^+$ because b_j is an upper bound for A in $\mathbb{R}$ by the claim. It follows that $\sup A \in I_j$ for each $j \in \mathbb{Z}^+$, so that $\sup A \in \bigcap_{j=1}^{\infty} I_j$, as desired. □

Definition 8.2. Let I be one of $[a,b], (a,b), (a,b], [a,b), (a,b)$ for some $a, b \in \mathbb{R}$ with $a \leq b$. Then the **length** of I is $b - a$.

Theorem 8.3 (Closed Intervals Are Compact in $\mathbb{R}$)
If $a, b \in \mathbb{R}$ with $a \leq b$, then $[a,b]$ is a compact subset of $\mathbb{R}$.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $[a,b]$ in $\mathbb{R}$. Suppose for the sake of contradiction that $[a,b]$ cannot be covered by finitely many U_α's. Let $L = \left[a, \frac{a+b}{2}\right]$ and $R = \left[\frac{a+b}{2}, b\right]$ so that $R \cup L = [a,b]$. Then it is not possible for both L and R to be covered by finitely many U_α's. Let $I_0 = [a,b]$, and let I_1 be L or R such that I_1 cannot be covered by finitely many U_α's. Repeating the process, we get an infinite sequence $I_0, I_1, I_2, I_3 \dots$ of closed intervals in $\mathbb{R}$ with the following properties

(1) $I_{i+1} \subseteq I_i$ for each $i \geq 0$

(2) $\text{length}(I_{i+1}) = \frac{1}{2}\text{length}(I_i)$ for each $i \geq 0$

(3) I_i cannot be covered by finitely many U_α's for each $i \geq 0$

By Proposition 8.1, $\bigcap_{i=0}^{\infty} I_i \neq \emptyset$. Let $x \in \bigcap_{i=0}^{\infty} I_i$. Since $x \in I_0 = [a, b] \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. And since U_{α_0} is an open set in $\mathbb{R}$, there is an $r > 0$ such that $(x - r, x + r) \subseteq U_{\alpha_0}$. By (2), we have that $\text{length}(I_i) = 2^{-i}(b - a)$ for every $i \geq 0$. It is easy to see that there is a nonnegative integer j such that $\text{length}(I_j) < r$.

Since $x \in I_j$, it follows that $I_j \subseteq (x - r, x + r)$, and so $I_j \subseteq U_{\alpha_0}$. This contradicts (3), as desired. □

§9 September 16, 2019

§9.1 Cells

Definition 9.1. Let $A_1, \dots, A_n$ be n sets, where n is a positive integer. Then their **Cartesian product** $A_1 \times \cdots \times A_n = \prod_{i=1}^{n} A_i$ is defined to be the set of n-tuples $(x_1, \dots, x_n)$ with $x_i \in A_i$ for each $i = 1, \dots, n$.

Definition 9.2. Let $I_i = [a_i, b_i]$ be a closed interval in $\mathbb{R}$ for each $i = 1, \dots, n$. Then their Cartesian product $C = \prod_{i=1}^{n} I_i$ is called a **cell** in $\mathbb{R}^n$.

Definition 9.3. The **diameter** of the cell $C = \prod_{i=1}^{n}[a_i, b_i]$ is defined to be $\operatorname{diam} C = \sqrt{\sum_{i=1}^{n}(b_i - a_i)^2}$, which is the maximum of the distances between elements in C with respect to the standard Euclidean metric in $\mathbb{R}^n$.

Proposition 9.4 (Intersection of Infinitely Nested Cells Is Nonempty)
If $C_1, C_2, C_3, \dots$ is an infinite sequence of cells in $\mathbb{R}^n$ such that $C_{i+1} \subseteq C_i$ for each $i \in \mathbb{Z}^+$, then $\bigcap_{i=1}^{n} C_i = \emptyset$.

Proof. For each $i \in \mathbb{Z}^+$, there are n closed intervals $I_{i,\ell}$, $1 \leq \ell \leq n$ in $\mathbb{R}$ such that $C_i = \prod_{\ell=1}^{n} I_{i,\ell}$, by definition of a cell in $\mathbb{R}^n$. It is easy to see that $I_{i+1,\ell} \subseteq I_{i,\ell}$ for every $i \in \mathbb{Z}^+$ and $\ell = 1, \dots, n$ because $C_{i+1} \subseteq C_i$. Note that $\bigcap_{i=1}^{\infty} I_{i,\ell} \neq \emptyset$ for each $\ell = 1, \dots, n$

by Proposition 8.1. Let x_ℓ be an element of $\bigcap_{i=1}^{\infty} I_{i,\ell}$ for each $\ell = 1, \ldots, n$. Then it is easy to see that $x = (x_1, \ldots, x_n) \in \bigcap_{i=1}^{n} C_i$, as desired. □

Proposition 9.5 (Partitioning Cells Into Smaller Cells)
If C is any cell in $\mathbb{R}^n$, then C can be expressed as the union of 2^n cells, each of which has diameter $\frac{1}{2} \operatorname{diam} C$.

Proof. By definition of a cell, there are n closed intervals $I_i = [a_i, b_i]$ with $1 \le i \le n$ in $\mathbb{R}$ such that $C = \prod_{i=1}^{n} I_i$. Let $L_i = [a_i, \frac{a_i + b_i}{2}]$ and $R_i = [\frac{a_i + b_i}{2}, b_i]$ for each $i = 1, \ldots, n$ so that $I_i = L_i \cup R_i$. Note that $\operatorname{length}(L_i) = \operatorname{length}(R_i) = \frac{1}{2}\operatorname{length}(I_i)$.

$$C = \prod_{i=1}^{n} I_i = \prod_{i=1}^{n} (L_i \cup R_i).$$

Consider the 2^n cells of the form $\prod_{i=1}^{n} K_i$, where for each $i = 1, \ldots, n$, K_i is either L_i or R_i. It is easy to see that C is the union of these 2^n cells, each of which has diameter equal to $\frac{1}{2} \operatorname{diam}(C)$. □

Theorem 9.6 ($\mathbb{R}^n$ Cells Are Compact)
The cell C in $\mathbb{R}^n$ is compact with respect to the standard Euclidean metric.

Sketch of Proof. This can be shown the same way as the case $n = 1$ in Theorem 8.3, using Proposition 9.4 and Proposition 9.5. □

Corollary 9.7 (Closed and Bounded Implies Compactness)
If $E \subseteq \mathbb{R}^n$ is closed and bounded, then E is compact with respect to the standard Euclidean metric.

Proof. Because E is bounded in $\mathbb{R}^n$, it is easy to show that there is a cell $C \subseteq \mathbb{R}^n$ such that $E \subseteq C$. By Theorem 8.3, C is compact in $\mathbb{R}^n$. By Proposition 6.1, it follows that E is compact because E is a closed set contained in C. □

Definition 9.8. A subset E of $\mathcal{M}$ is **totally bounded** in $\mathcal{M}$ if for each $r > 0$, there are finitely many elements $x_i, \ldots, x_n$ of $\mathcal{M}$ such that $E \subseteq \bigcup_{i=1}^{n} B(x_i, r)$.

If $E = \mathcal{M} = \emptyset$, then this may be considered to hold for $n = 0$.

§10 September 18, 2019

Proposition 10.1 (Compactness Implies Total Boundedness)
If $K \subseteq \mathcal{M}$ is compact, then K is totally bounded.

Proof. Let $r > 0$. Observe that $K \subseteq \bigcup_{x \in K} B(x, r)$ because $x \in B(x, r)$ for every $x \in K$. It follows that the collection of open balls $B(x, r)$ with $x \in K$ is an open covering of K in $\mathcal{M}$. Since K is compact, this open covering can be reduced to a finite subcovering consisting of $B(x_i, r)$ for $i = 1, \ldots, n$. So $K \subseteq \bigcup_{i=1}^{n} B(x_i, r)$, as desired. □

Proposition 10.2 (Limit Point Property Implies Total Boundedness)
If $E \subseteq \mathcal{M}$ has the limit point property, then E is totally bounded.

Proof. Let $r > 0$. If $E = \emptyset$, then we are done. Otherwise, let $x_1 \in E$. If $E \subseteq B(x, r)$, then we are done. Otherwise, if $E \not\subseteq B(x, r)$, then let $x_2 \in (E \setminus B(x_1, r))$. We describe the general x_i as follows: if $x_1, \ldots, x_n \in E$ have been chosen for some $n \in \mathbb{Z}^+$. If $E \subseteq \bigcup_{i=1}^{n} B(x_i, r)$, then we are done. Otherwise, let $x_{n+1} \in E \setminus \bigcup_{i=1}^{n} B(x_i, r)$.

If the process terminates in a finite number of steps, then we are done. Otherwise, suppose this process never terminates in finitely many steps. Then we get an infinite sequence $x_1, x_2, \ldots$ of elements of E such that $d(x_i, x_j) \geq r$ for $i \neq j$. Let L be the set of x_i's, $i \in \mathbb{Z}^+$. Note that L has infinitely many elements since $x_i \neq x_j$ for $i \neq j$. Since E has the limit point property and $L \subseteq E$ is infinite, so there must be a point $p \in E$ that is a limit point of L in $\mathcal{M}$. Thus, there must be infinitely many elements of L in $B(p, \frac{r}{2})$ by Proposition 2.7. Let $x_i, x_j \in B(p, \frac{r}{2})$ with $i < j$. By the triangle inequality, $d(x_i, x_j) \leq d(x_i, p) + d(p, x_j) < \frac{r}{2} + \frac{r}{2} = r$, which is a contradiction. □

§10.1 Countability of Sets

Definition 10.3. A set A is **countably infinite** if there is an infinite sequence $\{a_i\}_{i=1}$ of elements of A in which elements of A occurs exactly once, i.e. for each $a \in A$, there is a unique $i \in \mathbb{Z}^+$ such that $A_i = a$

Example 10.4 (Countably Infinite Sets)
The following sets are countably infinite.

1. The set $\mathbb{Z}^+$ of positive integers is countably infinite since we can take $a_i = i$ for each $i \in \mathbb{Z}^+$.
2. The set $\mathbb{Z}$ of all integers is countably finite. Indeed, we can let $x_i = \frac{i}{2}$ when i is even and $x_i = -\frac{i-1}{2}$ when i is odd.

Proposition 10.5

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of elements of a set B such that every element of a set A occurs in the sequence at least once. Then A has finitely or countably many elements.

Proof. If $A = \emptyset$, then we are done. Otherwise, let i_1 be the smallest positive integer such that $x_{i_1} \in A$. If $A = \{x_1\}$, then we are done. Otherwise, if $x_1, \ldots, x_n$, $n \in \mathbb{Z}^+$, have been chosen in this way. If $A = \{x_1, \ldots, x_n\})$, then we are done. Otherwise, let $x_{i_{n+1}}$ be the smallest positive integer such that $x_{i_{n+1}} \in A \setminus \{x_1, \ldots, x_n\}$. If the process stops after finitely many steps, then A has finitely many elements. Otherwise, we get an infinite sequence of A such that every element of A occurs exactly once, as desired. □

Corollary 10.6 (Subsets of Countable Sets Are Countable)

If B is a countably infinite set, and $A \subseteq B$, then A has finitely or countably manny elements.

§11 September 23, 2019

Lemma 11.1

If $A_1, A_2, \ldots$ is an infinite sequence of finite sets, then $\bigcup_{i=1}^{\infty} A_i$ has only finitely or countably many elements.

Proof. We can get a sequence in which every element of $\bigcup_{i=1}^{\infty} A_i$ occurs at least once by listing elements of A first, then A_2, and so on. □

Example 11.2 (Nonnegative Ordered Pairs Are Countable)

$\mathbb{Z}_+^2 = \mathbb{Z}^2 \times \mathbb{Z}^+$ is countably infinite.

Proof. Put $A_n = \{(i, j) \in \mathbb{Z}_+^2 \mid i + j = n + 1\}$ for every $n \in \mathbb{Z}^+$. It is easy to see that A_n has exactly n elements for each $n \in \mathbb{Z}^+$ and that $\mathbb{Z}_+^2 = \bigcup_{n=1}^{\infty} A_n$. Lemma 11.1 implies that $\mathbb{Z}_+^2$ has only finitely or countably many elements, and hence that $\mathbb{Z}_+^2$ is countably infinite since $\mathbb{Z}_+^2$ is obviously infinite. □

Proposition 11.3 (Union of Countable Sets Is Countable)

If $A_1, A_2, \ldots$ is an infinite sequence of sets, each of which has only finitely or countably many elements, then $\bigcup_{i=1}^{\infty} A_i$ has only finitely or countably many elements.

Proof. By hypothesis, for each $j \in \mathbb{Z}^+$, there is a sequence $\{x_{i,j}\}_{i=1}$ in which every element of A_j occurs at least once. By the countability of $\mathbb{Z}_+^2$ in Example 11.2, there is a sequence $\{(i_n, j_n)\}_{n=1}$ of elements of $\mathbb{Z}_+^2$ in which every element of $\mathbb{Z}_+^2$ occurs once. Using this, we get a sequence $\{x_{i_n,j_n}\}_{n=1}$ in which every element of $\bigcup_{j=1}^{\infty} A_j$ occurs at least once, as desired. □

Example 11.4 (Rationals Are Countable)
The set $\mathbb{Q}$ of rational numbers is countably infinite.

Proof. Indeed, for each $j \in \mathbb{Z}^+$, let A be the set of rational numbers of the form $\frac{i}{j}$, where $i \in \mathbb{Z}$. It is easy to see that A_j is countably infinite for each $j \in \mathbb{Z}^+$ because $\mathbb{Z}$ is countably infinite. It follows that $\mathbb{Q} = \bigcup_{j=1}^{\infty} A_j$ has only finitely or countably many elements. Of course, $\mathbb{Q}$ is an infinite set because $\mathbb{Z}$ is infinite, and hence $\mathbb{Q}$ is countably infinite. □

Proposition 11.5

Let $A_1, \ldots, A_n$ be countably infinite sets for $n \in \mathbb{Z}^+$. Then $\prod_{i=1}^{n} A_i$ is countably infinite.

Proof. We proceed by induction on n. The case $n = 1$ is trivial. The case $n = 2$ can be obtained from the countability of $\mathbb{Z}_+^2$ in Example 11.2. Suppose now that $A_1, \ldots, A_n$ are n countably infinite sets for some $n > 1$. By the inductive hypothesis, $\prod_{i=1}^{n-1} A_i$ is countably infinite. Then we can reduce $\prod_{i=1}^{n} A_i = \left(\prod_{i=1}^{n-1}\right) \times A_n$ to the case when $n = 2$, which completes the proof.

Alternatively, we can consider $\prod_{i=1}^{n} A_i$ as the countable union of the sets $\left(\prod_{i=1}^{n-1}\right) \times \{a_n\}$ with $a_n \in A_n$. □

Definition 11.6. A set A is **uncountable** if A is neither finite nor countably infinite.

Example 11.7 (Set of All Binary Sequences Is Uncountable)
Let B be the set of all infinite sequences $x = \{x_i\}_{i=1}^{\infty}$ such that for each $i \in \mathbb{Z}^+$, $x_i \in \{0, 1\}$.

Proof. Let $x(1) = \{x_i(1)\}_{i=1}^{\infty}, x(2), x(3), \ldots$ be any sequence of elements of B. For each $i \in \mathbb{Z}^+$, let $y_i = \neg x_i$. Thus, $y = \{y_i\}_{i=1}^{\infty} \in B$, and for each $j \in \mathbb{Z}^+$, $y \neq x(j)$ because $y_j \neq x_j(j)$. This implies that there is no sequence of elements of B in which every element of B occurs, so B must be uncountably infinite, as desired. □

Remark 11.8. We can use the uncountability of the set B to get that $[0,1]$ is uncountable, and hence that $\mathbb{R}$ is uncountable.

§12 September 25, 2019

§12.1 Density

Definition 12.1. A subset E of $\mathcal{M}$ is **dense** in $\mathcal{M}$ if every element of $\mathcal{M}$ is an element or a limit point of E.

Remark 12.2. This is equivalent to saying $\bar{E} = \mathcal{M}$.

Definition 12.3. $\mathcal{M}$ is **separable** as a metric space if there is a dense subset E of $\mathcal{M}$ such that E has finitely or countably many elements.

Example 12.4 (Separable Metric Spaces)
The following are separable metric spaces:

1. Any finite or countable set $\mathcal{M}$ with respect to any metric.
2. $\mathbb{R}^n$ with respect to the standard Euclidean metric.

The first example is trivial. For the second example, consider $\mathbb{Q}^n$ to be the set of n-tuples of rational numbers. Then $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ with respect to the standard Euclidean metric and has countably infinite elements.

Proposition 12.5 (Separable Spaces Can Be Separated)
$\mathcal{M}$ is separable if and only if for each $r > 0$, there is a subset E_r of $\mathcal{M}$ such that E_r has only finitely or countably many elements, and $\displaystyle\bigcup_{x \in E(r)} B(x, r) = \mathcal{M}$.

Proof. We first show the reverse direction. For each $i \in \mathbb{Z}^+$, let $E(\frac{1}{i})$ be as in the proposition with $r = \frac{1}{i}$. Let $E = \displaystyle\bigcup_{i=1}^{n} E(\frac{1}{i})$. Then E is finite or countably infinite by Proposition 11.3. It is easy to see that E is dense in $\mathcal{M}$ because $\displaystyle\bigcup_{x \in E(\frac{1}{i})} B(x, \frac{1}{i}) = \mathcal{M}$ for each $i \in \mathbb{Z}^+$. This concludes the proof for the reverse direction.

We now show the forward direction. Suppose $\mathcal{M}$ is separable and let E be a dense subset of $\mathcal{M}$ with only finitely or countably many elements. If r is a positive real number, then it is easy to see that $E_r = E$ has the properties stated in the proposition. □

Corollary 12.6 (Totally Boundedness Implies Separability)
If $\mathcal{M}$ is totally bounded, then $\mathcal{M}$ is separable.

Proof. For each $r > 0$, there is a finite set E_r such that $\displaystyle\bigcup_{x \in E_r} B(x, r) = \mathcal{M}$. □

Definition 12.7. Let B be a collection of open subsets of $\mathcal{M}$. Then B is a **base for the topology** of $\mathcal{M}$ if for every $x \in \mathcal{M}$ and $r > 0$, there is a $V \in B$ such that $x \in V$ and $V \subseteq B(x, r)$.

Example 12.8 (Topology Bases)
The following are bases for the topology of $\mathcal{M}$.

1. The collection of all open balls in $\mathcal{M}$.
2. The collection of all open balls in $\mathcal{M}$ of the form $B(x, \frac{1}{n})$ with $x \in \mathcal{M}$ and $n \in \mathbb{Z}^+$.
3. The collection of open balls in $\mathcal{M}$ of the form $B(x, \frac{1}{n})$ with $x \in E$, where E is a dense subset of $\mathcal{M}$ and $n \in \mathbb{Z}^+$.

§13 September 27, 2019

Proposition 13.1 (Separability And Existence of Countable Base Are Equivalent)
$\mathcal{M}$ is separable if and only if there is a countable base B for the topology of $\mathcal{M}$.

Proof. Suppose $\mathcal{M}$ is separable, so that there is a dense set $E \subseteq \mathcal{M}$ with only finitely or countably many elements. Let B be the collection of open subsets of $\mathcal{M}$ of the form $B(x, \frac{1}{i})$, where $x \in E$ and $i \in \mathbb{Z}^+$. This defines a base for the topology of $\mathcal{M}$. For each $i \in \mathbb{Z}^+$, let B_i be the collection of open subsets of $\mathcal{M}$ of the form $B(x, \frac{1}{i})$ with $x \in E$. It is easy to see that B_i has only finitely or countably many elements for each $i \in \mathbb{Z}^+$ because E has only finitely or countably many elements. Observe that $B = \bigcup_{i=1}^{\infty} B_i$ by definition of B and B_i. It follows that B has only finitely or countably many elements by Proposition 11.3, as desired.

Suppose that B is a base for the topology of $\mathcal{M}$ with only finitely or countably many elements. For each $V \in B$ with $V \neq \emptyset$, pick a point $X_V \in V$. Let E be the set of points $X_V, V \in B, V \neq \emptyset$, that have been chosen this way. It is easy to see that E has only finitely or countably many elements because B has only finitely or countably many elements. Since B is a base for the topology of $\mathcal{M}$, then E is dense in $\mathcal{M}$, as desired. ☐

Theorem 13.2 (Lindelöf's Theorem)
If B is a countable base for the topology of $\mathcal{M}$, and $\{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of $\mathcal{M}$, then there is a countable subset A_1 of A such that

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha.$$

Proof. For each $\alpha \in A$, let $B_\alpha = \{V \in B \mid V \subseteq U_\alpha\}$. Note that $\bigcup_{V \in B_\alpha} V \subseteq U_\alpha$ for each $\alpha \in A$, by definition of B_α. Then it is easy to see that $\bigcup_{V \in B_\alpha} V = U_\alpha$ for every $\alpha \in A$ because B is a base for the topology of $\mathcal{M}$. Let $\tilde{B} = \bigcup_{\alpha \in A} B_\alpha$. Observe that $\bigcup_{V \in \tilde{B}} V = \bigcup_{\alpha \in A} \left(\bigcup_{V \in B_\alpha} V \right) = \bigcup_{\alpha \in A} U_\alpha$.

For each $v \in \tilde{B}$, let $\alpha_v \in A$ such that $v \in B_{\alpha_v}$, which is possible because $v \in \bigcup_{\alpha \in A} B_\alpha$. Note that $\tilde{B}$ is countable because $\tilde{B} \subseteq B$. Let A_1 be the set of $\alpha_v, v \in \tilde{B}$. It is easy to see that A_1 is countable because $\tilde{B}$ is also countable. Observe that $\bigcup_{\alpha \in A} U_\alpha = \bigcup_{v \in \tilde{B}} v \subseteq \bigcup_{v \in \tilde{B}} U_{\alpha_v}$ because for each $v \in \tilde{B}$, $v \in B_{\alpha_v}$ and hence $v \subseteq U_{\alpha_v}$. We also have that $\bigcup_{v \in \tilde{B}} U_{\alpha_v} = \bigcup_{\alpha \in A_1} U_\alpha$ by definition of A. Combining the two results yields $\bigcup_{\alpha \in A} U_\alpha \subseteq U_{\alpha \in A_1} U_\alpha$. This implies that $\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha$ because $A_1 \subseteq A$, as desired. □

§14 September 30, 2019

Proposition 14.1 (Limit Point Property Implies Weak Compactness)
Suppose $E \subseteq \mathcal{M}$ has the limit point property. If $U_1, U_2, U_3, \ldots$ is an infinite sequence of open subsets of $\mathcal{M}$ such that $E \subseteq \bigcup_{i=1}^{\infty} U_i$, then there is a positive integer n such that $E \subseteq \bigcup_{i=1}^{n} U_i$.

Proof. Suppose for the sake of contradiction that for each $n \in \mathbb{Z}^+$, $E \not\subseteq \bigcup_{i=1}^{n} U_i$. For each $\ell \in \mathbb{Z}^+$, let $x_\ell \in E \setminus \left(\bigcup_{i=1}^{\ell} U_i \right) \neq \emptyset$. Let L be the set of points $x_\ell, \ell \in \mathbb{Z}^+$ that have been chosen in this way.

We claim that L has infinitely many elements. Suppose for the sake of contradiction that L has only finitely many elements. This implies that there is an $x \in E$ such that $x = x_\ell$ for infinitely many $\ell \in \mathbb{Z}^+$. Because $x \in E \subseteq \bigcup_{i=1}^{\infty} U_i$, then there is an $i \in \mathbb{Z}^+$ such that $x \subseteq U_i$. This implies that $x \neq x_\ell$ when $\ell \geq i$, since $x_\ell \in E \setminus \left(\bigcup_{i=1}^{\ell} U_i \right)$ for each $\ell \in \mathbb{Z}^+$. Thus, $x = x_\ell$ for at most finitely many $\ell \in \mathbb{Z}^+$, which is a contradiction, as desired. So L is indeed infinite.

If E has the limit point property, then it follows that there is a point $p \in E$ such that p is a limit point of L in $\mathcal{M}$. Because $p \in E \subseteq \bigcup_{i=1}^{\infty} U_i$, then there is a positive integer j such that $p \in U_j$. Because U_j is an open set in $\mathcal{M}$, there is an $r > 0$ such that $B(p, r) \subseteq U_j$. Since p is a limit point of L in $\mathcal{M}$, then there are infinitely many elements of L in $B(p, r)$

by Proposition 2.7. This implies that $x_\ell \in B(p,r) \subseteq U_j$ for infinitely many $\ell \in \mathbb{Z}^+$. However, if $\ell \geq j$, then $x_\ell \in E \setminus \left(\bigcup_{i=1}^{\ell} U_i\right) \supseteq E \setminus U_j \supseteq E \setminus B(p,r)$ by construction. Thus, $x_\ell \in B(p,r) \subseteq U_j$ can only happen when $\ell < j$, and in particular, this can only happen for finitely many $\ell \in \mathbb{Z}^+$, as desired. □

§15 October 2, 2019

Theorem 15.1 (Limit Point Property Implies Compactness)
If $E \subseteq \mathcal{M}$ has the limit point property, then E is compact.

Proof. Suppose first that $E = \mathcal{M}$, so that $\mathcal{M}$ has the limit point property. This implies that $\mathcal{M}$ is totally bounded, and hence separable, and also that there is a base for the topology of $\mathcal{M}$ with only finitely or countably many elements. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $\mathcal{M}$. By Lindelöf's Theorem, there is a subset A_1 of A such that A_1 has only finitely or countably many elements and $\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha = \mathcal{M}$. If A_1 has only finitely many elements, then there is nothing more to do. Otherwise, if A_1 is countably infinite, then Proposition 14.1 implies that there is a finite subset A_2 of A_1 such that $\mathcal{M} \subseteq \bigcup_{\alpha \in A_2} U_\alpha$, as desired.

Now let E be any subset of $\mathcal{M}$ with the limit point property. Note that E can also be considered as a metric space with respect to the restriction of $d(x,y)$ to $x, y \in E$. It is easy to see that E has the limit point property as a subset of itself because E has the limit point property as a subset of $\mathcal{M}$. By the above proof, we obtain that E is compact as a subset of itself. By Proposition 7.1, this implies that E is compact as a subset of $\mathcal{M}$, as desired. □

§15.1 Connected Sets

Definition 15.2. Two subsets A and B are **separated** in $\mathcal{M}$ if $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Definition 15.3. A subset E of $\mathcal{M}$ is **connected** in $\mathcal{M}$ if E cannot be expressed as the union of two nonempty separated subsets of $\mathcal{M}$.

Example 15.4
Take $\mathcal{M} = \mathbb{R}$ with the standard Euclidean metric.

1. $[0,1) \cup (1,2)$ is not connected.
2. If $A = [0,1]$ and $B = (1,2)$, then $A \cap B = \emptyset$ but $A \cap \bar{B} = \{1\} \neq \emptyset$.

Theorem 15.5
A subset E of $\mathbb{R}$ is connected if and only if E has the following property: if $x, y \in E, z \in \mathbb{R}$, and $x < z < y$, then $z \in E$.

Proof. Suppose that the property does not hold. We will show that E is not connected. Because the property does not hold, there are $x, y \in E$ and $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin E$. Let $A = \{w \in E \mid w < z\}$ and $B = \{w \in E \mid w > z\}$.Note that $E = A \cup B$ because $z \notin E$. We also have that A and B are nonempty because $x \in A, y \in B$ by construction. It is easy to see that $\bar{A} \subseteq \{w \in \mathbb{R} \mid w \leq z\}$ and $\bar{B} \subseteq \{w \in \mathbb{R} \mid w \geq z\}$. It follows that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ by definition of A and B. This implies that E is not connected, as desired.

Suppose now that E satisfies the property, and for the sake of contradiction, that E is not connected. Because E is connected, there are nonempty separated sets $A, B \subseteq \mathbb{R}$ such that $A \cup B = E$. Let x be an element of A and let y be an element of B, which exist since $A, B \neq \emptyset$. Without loss of generality, we may suppose that $x < y$ since $x \neq y$ from A, B being disjoint and we can just interchange A and B if $x > y$. Let $w = \sup(A \cap [x, y])$. Note that $A \cap [x, y] \neq \emptyset$ since $x \in A \cap [x, y]$ and that $w \geq x$. We also have that y is an upper bound for $A \cap [x, y]$ and hence $w \leq y$. Since $w \in \overline{A \cap [x, y]}$. This implies that $w \in \bar{A}$. Since A and B are separated in $\mathbb{R}$, then $\bar{A} \cap B = \emptyset$, implying that $w \notin B$. From the given property, $w \in E = A \cup B$ and hence $w \in A$. In particular, $w \neq y$. Using the property again, we get that $(w, y] \subseteq E = A \cup B$. We also have that $(w, y] \cap A = \emptyset$ by definition of w, so that $(w, y] \subseteq B$. This implies that $w \in \bar{B}$. It follows that $A \cap \bar{B} \neq \emptyset$ since $w \in A \cap \bar{B}$. This contradicts the assumption that A and B are separated in $\mathbb{R}$. □

§16 October 4, 2019

§16.1 Perfect Sets

Definition 16.1. A subset E of $\mathcal{M}$ is **perfect** if E is a closed set and every element of E is a limit point of E.

Theorem 16.2

If E is a nonempty perfect subset of $\mathbb{R}^n$ for some $n \in \mathbb{Z}^+$, then E is uncountable.

Proof. See Rudin's book. □

§16.2 Cantor Set

Definition 16.3. The **Cantor Set** is defined as $E = \bigcap_{i=0}^{\infty} E_i$, where

$$E_0 = [0, 1], \quad E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \quad E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we can get an infinite sequence $E_0, E_1, \ldots$ of subsets of $\mathbb{R}$ where for each $i \geq 0$, E_i is the union of 2^i pairwise disjoint closed intervals of length 3^{-i} in $\mathbb{R}$ and E_{i+1} is obtained from E_i by removing the open "middle third" from each of the 2^i intervals in E_i. We also get $E_{i+1} \subseteq E_i$.

Remark 16.4. E and E_i are closed sets in $\mathbb{R}$ for every $i \in \mathbb{Z}_{\geq 0}$.

Remark 16.5. The endpoints $0, 1 \in E_i$ for each $i \geq 0$, so that $0, 1 \in E$. Similarly, if A_i is the 2^{i+1} points in $\mathbb{R}$ that occur as endpoints of the 2^i intervals in E_i, then $A_i \subseteq E_i$ and $A_i \subseteq A_{j+1}$ for each $i \geq 0$. Then $A = \bigcup_{i=0}^{\infty} A_i$ is a subset of E.

Remark 16.6. $\bar{A} = E$.

Remark 16.7. Every element of E is a limit point of A.

Remark 16.8. E is perfect as a subset of $\mathbb{R}$.

Remark 16.9. E is uncountable. But A is countable, so $A \neq E$.

Remark 16.10. E does not contain any interval in $\mathbb{R}$ of positive length.

§17 October 7, 2019

§17.1 Complex Numbers

Definition 17.1. A **complex number** z can be expressed in a unique way as $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.

Definition 17.2. The **absolute value** or **modulus** of z is the nonnegative real number defined by $|z| = \sqrt{x^2 + y^2}$.

Remark 17.3. The definition of absolute values for complex numbers agrees with the usual absolute value of a real number.

Definition 17.4. The **complex conjugate** of z is defined by $\bar{z} = x - iy$.

Remark 17.5. If z and w are complex numbers, then it is easy to see that $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{(zw)} = \bar{z}\bar{w}$.

Remark 17.6. We also have that $z\bar{z} = |z|^2$. It follows that $|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2 |w|^2$, so that $|zw| = |z| |w|$.

Remark 17.7. $|z| > 0$ when $z \neq 0$. In this case $z\left(\frac{\bar{z}}{|z|^2}\right) = \frac{z\bar{z}}{|z|^2} = 1$, so that z has the **multiplicative inverse** $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

Definition 17.8. Let $\mathbb{C}$ be the set of complex numbers. We can identify $\mathbb{C}$ with $\mathbb{R}^2$ by identifying $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ with $(x, y) \in \mathbb{R}^2$.

Remark 17.9. $|z|$ corresponds to the standard Euclidean norm on $\mathbb{R}^2$. In particular, if $z, w \in \mathbb{C}$, then $|z + w| \leq |z| + |w|$.

Remark 17.10. We can define the standard Euclidean metric metric on $\mathbb{C}$ by $d(z, w) = |z - w|$.

§17.2 Convergence

Definition 17.11. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ **converges** to an element x of $\mathcal{M}$ if for every $\epsilon > 0$, there is an $L \in \mathbb{Z}^+$ such that for every $i \in \mathbb{Z}^+$ with $i \geq L$, we have that $d(x_i, x) < \epsilon$.

Proposition 17.12 (Uniqueness of Convergence)
If $\{x_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathcal{M}$ that converges to elements x and x' of $\mathcal{M}$, then $x = x'$.

Proof. Suppose for the sake of contradiction that $x \neq x'$. Thus, $d(x, x') > 0$. Let $\epsilon = \dfrac{d(x, x')}{2} > 0$. Because $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{M}$, there is an $L \in \mathbb{Z}^+$ such that for every $i \in \mathbb{Z}^+$, with $i \geq L$, we have that $d(x_i, x) < \epsilon$. Similarly, because $\{x_i\}_{i=1}^{\infty}$ converges to x' in $\mathcal{M}$, there is an $L' \in \mathbb{Z}^+$ such that for every $i \in \mathbb{Z}^+$, $d(x_i, x') < \epsilon$. It follows that for every $i \in \mathbb{Z}^+$ with $i \geq \max(L, L')$, we have that $d(x, x') \leq d(x, x_i) + d(x_i, x') < \epsilon + \epsilon = d(x, x')$. In particular, this holds when $i = \max(L, L')$, which is a contradiction. □

Definition 17.13. If $\{x_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathcal{M}$ that converges to an element x of $\mathcal{M}$, then we call x the **limit** of $\{x_i\}_{i=1}^{\infty}$ and express this by $\lim_{i\to\infty} x_i = x$.

Definition 17.14. We may also say that x_i **tends** to x as $i \to \infty$, or $x_i \to \infty$ as $i \to \infty$.

Definition 17.15. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ is **bounded** in $\mathcal{M}$ if the set of x_i's ($i \in \mathbb{Z}^+$) is bounded in $\mathcal{M}$.

Proposition 17.16 (Convergence Implies Boundedness)

If a sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ converges to an element x of $\mathcal{M}$, then $\{x_i\}_{i=1}^{\infty}$ is bounded in $\mathcal{M}$.

Proof. Because $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{M}$, there is an $L \in \mathbb{Z}^+$ such that for every $i \in \mathbb{Z}^+, i \geq L$, we have that $d(x_i, x) < 1$. If $L > 1$, then let $R = \max_{1 \leq i \leq L} d(x_i, x)$, and let $R = 0$ otherwise when $L = 1$. It follows that $d(x_i, x) < R + 1$ for every $i \in \mathbb{Z}^+$, so that $x_i \in B(x, R+1)$, as desired. □

§18 October 9, 2019

Proposition 18.1 (Sum of Convergent Sequences)

If $\{z_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ are sequences of complex numbers that converge to $z, w \in \mathbb{C}$, respectively, then $\{z_i + w_i\}_{i=1}^{\infty}$ converges to $z + w$ in $\mathbb{C}$.

Proof. Let $\epsilon > 0$ be given. Since $\{z_i\}$ converges to z in $\mathbb{C}$, there is an $L \in \mathbb{Z}^+$ such that $|z - z_i| < \dfrac{\epsilon}{2}$ for every $i \geq L$. Similarly, since $\{w\}$ converges to w in $\mathbb{C}$, there is an $L' \in \mathbb{Z}^+$ such that $|w - w_i| < \dfrac{\epsilon}{2}$ for every $i \geq L'$. Observe that $|(z + w) - (z_i + w_i)| = |(z - z_i) + (w - w_i)| \leq |z - z_i| + |w - w_i|$ for every $i \in \mathbb{Z}^+$. If $i \geq \max(L, L')$, then it follows that $|(z + w) - (z_i + w_i)| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$, as desired. □

Lemma 18.2 (Scalar Multiple of Convergent Sequence)

If $\{z_i\}_{i=1}^{\infty}$ is a sequence of complex numbers that converges to $z \in \mathbb{C}$ and $c \in \mathbb{C}$, then $\{cz_i\}_{i=1}^{\infty}$ converges to cz in $\mathbb{C}$.

Proof. If $c = 0$, then $cz_i = 0$ for every $i \in \mathbb{Z}^+$, and so $\{cz_i\}_{i=1}^{\infty}$ converges to $cz = 0$ in $\mathbb{C}$.

Suppose now that $c \neq 0$, and let $\epsilon > 0$ be given. Since $\{z_i\}_{i=1}^{\infty}$ converges to z in $\mathbb{C}$, there is an $L \in \mathbb{Z}^+$ such that $|z - z_i| < \dfrac{\epsilon}{|c|}$ for every $i \geq L$. It follows that $|cz - cz_i| = |c(z - z_i)| = |c|\,|z - z_i| < |c|\left(\dfrac{\epsilon}{|c|}\right) = \epsilon$ for every $i \geq L$, as desired. □

Lemma 18.3 (Product of Zero Convergent and Bounded Sequence)

If $\{a_i\}_{i=1}^{\infty}$ is a sequence of complex numbers that converges to 0 in $\mathbb{C}$, and if $\{b_i\}_{i=1}^{\infty}$ is a bounded sequence of complex numbers, then $\{a_i b_i\}_{i=1}^{\infty}$ converges to 0 in $\mathbb{C}$.

Proof. Because $\{b_i\}_{i=1}^{\infty}$ is a bounded sequence in $\mathbb{C}$, there is a positive real number B such that $|b_i| \leq B$ for every $i \in \mathbb{Z}^+$. Let $\epsilon > 0$ be given. Because $\{a_i\}_{i=1}^{\infty}$ converges to 0 in $\mathbb{C}$, there is an $L \in \mathbb{Z}^+$ such that $|a_j| < \dfrac{\epsilon}{B}$ for every $i \geq L$. It follows that $|a_i b_i| = |a_i|\,|b_i| \leq |a_i|\,B < \dfrac{\epsilon}{B} \cdot B = \epsilon$, as desired. □

Proposition 18.4 (Product of Convergent Sequences)

If $\{z_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ are sequences of complex numbers that converge to $z, w \in \mathbb{C}$, respectively, then $\{z_i w_i\}_{i=1}^{\infty}$ converges to zw in $\mathbb{C}$.

Proof. It is easy to see that $\{z_i - z\}_{i=1}^{\infty}$ converges to 0 in $\mathbb{C}$. Note that $\{w_i\}_{i=1}^{\infty}$ is bounded in $\mathbb{C}$ by Proposition 17.16 because it converges in $\mathbb{C}$. It follows that $\{(z_i - z)w_i\}_{i=1}^{\infty}$ converges to 0 in $\mathbb{C}$ by Lemma 18.3. By Lemma 18.2, we have that $\{zw_i\}_{i=1}^{\infty}$ converges to zw in $\mathbb{C}$. Hence $z_i w_i = (z_i - z)w_i + zw_i \to 0 + zw = zw$ as $i \to \infty$ by Proposition 18.1. □

§19 October 11, 2019

Proposition 19.1 (Reciprocal of Convergent Sequences)

If $\{z_i\}_{i=1}^{\infty}$ is a sequence of nonzero complex numbers that converges to a nonzero complex number z, then $\{\dfrac{1}{z_i}\}_{i=1}^{\infty}$ converges to $\dfrac{1}{z}$ in $\mathbb{C}$.

Proof. Observe that $\left|\dfrac{1}{z} - \dfrac{1}{z_i}\right| = \dfrac{z_i - z}{zz_i}$ for every $i \in \mathbb{Z}^+$. Hence, $\left|\dfrac{1}{z} - \dfrac{1}{z_i}\right| = \dfrac{|z_i - z|}{|z|\,|z_i|}$ for every $i \in \mathbb{Z}^+$ (Δ). Because the sequence $\{z_i\}_{i=1}^{\infty}$ converges to z, there is an $L_0 \in \mathbb{Z}^+$ such that $|z - z_i| < \dfrac{|z|}{2}$ for every $i \geq L_0$. If $i \geq L_0$, then it follows that $|z| \leq |z - z_i| + |z_i| < \dfrac{|z|}{2} + |z_i|$ using the triangle inequality. This implies that $|z_i| > |z| - \dfrac{|z|}{2} = \dfrac{|z|}{2}$ when $i \geq L_0$. Combining this with Δ, we get that $\left|\dfrac{1}{z} - \dfrac{1}{z_i}\right| = \dfrac{|z_i - z|}{|z|\,|z_i|} \leq \dfrac{2}{|z|^2}|z_i - z|$ for every $i \geq L_0$. Let $\epsilon > 0$ be given. Using the convergence of $\{z_i\}_{i=1}^{\infty}$ to z again, we get that there is an $L \in \mathbb{Z}^+$ such that $|z_i - z| < \dfrac{|z|^2}{2}$ for every $i \geq L$. If $i \geq \max(L_0, L)$, then it follows that $\left|\dfrac{1}{z} - \dfrac{1}{z_i}\right| \leq \dfrac{2}{|z|^2}|z_i - z| < \dfrac{2}{|z|^2}\left(\dfrac{|z|^2}{2}\epsilon\right) = \epsilon$, as desired. □

Proposition 19.2 (Bounded Increasing Sequences Converge)

If $\{x_i\}_{i=1}^{\infty}$ is a monotonically increasing sequence of real numbers that has an upper bound in $\mathbb{R}$, then the sequence converges in $\mathbb{R}$.

Proof. Let $A = \{x_i\}_{i=1}^{\infty}$. Since A is nonempty (as it contains x_1) and has an upper bound in $\mathbb{R}$, then the supremum of A exists in $\mathbb{R}$. Let $x = \sup A$. We will show that A converges to x in $\mathbb{R}$. Let $\epsilon > 0$ be given. Observe that $x - \epsilon$ is not an upper bound for A because $x - \epsilon < x = \sup A$. Then there is a $j \in \mathbb{Z}^+$ such that $x_j > x - \epsilon$. If $i \geq j, i \in \mathbb{Z}^+$, then $x_i \geq x_j > x - \epsilon$ since A is monotonically increasing. Of course, $x_i \leq x$ for every $i \in \mathbb{Z}^+$ by definition of x. Thus, $x - \epsilon < x \leq x$ for every $i \geq j$ so that $|x - x_i| = x - x_i < \epsilon$ for every $i \geq 0$, as desired. □

Corollary 19.3 (Bounded Decreasing Sequences Converge)

If $\{x_i\}_{i=1}^{\infty}$ is a monotonically decreasing sequence of real numbers that has a lower bound in $\mathbb{R}$, then the sequence converges in $\mathbb{R}$.

Proof. Similar to above. □

Definition 19.4. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of elements of some set and $\{i_\ell\}_{\ell=1}^{\infty}$ be a sequence of strictly increasing sequence of positive integers so that $i_\ell < i_{\ell+1}$ for each $\ell \in \mathbb{Z}^+$. Then $\{x_{i_\ell}\}_{\ell=1}^{\infty}$ is a **subsequence** of $\{x_i\}_{i=1}^{\infty}$.

Remark 19.5. $\{x_i\}_{i=1}^{\infty}$ is a subsequence of itself.

Remark 19.6. If a sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ converges to an element x of $\mathcal{M}$, then every subsequence of $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{M}$.

Definition 19.7. A subset K of $\mathcal{M}$ is **sequentially compact** if every sequence of elements of K has a subsequence that converges to an element of K in $\mathcal{M}$.

Proposition 19.8 (Sequential Compactness $\iff$ Limit Point Property)

A subset K of $\mathcal{M}$ is sequentially compact if and only if K has the limit point property.

Proof. Suppose K has the limit point property. Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary sequence of elements of K. Consider the set X of x_i's, $i \in \mathbb{Z}^+$. If X has finitely many elements, then there is an $x \in K$ such that $x_o = x$ for infinitely many $i \in \mathbb{Z}^+$. This implies that there is a subsequence $\{x_{i_\ell}\}_{\ell=0}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that $x_{i_\ell} = x$ for every $\ell \in \mathbb{Z}^+$. It follows that this subsequence converges to x in $\mathcal{M}$, as desired. If X has infinitely many elements, then the limit point property for K implies that there is an element x of K such that x is a limit point of X in $\mathcal{M}$. This implies that for each $r > 0$, there are infinitely many elements of L in $B(x, r)$ by Proposition 2.7 (Δ). It follows that for each $r > 0$, there are infinitely many $i \in \mathbb{Z}^+$ such that $d(x, x_i) < r$. In particular, there is an $i_i \in \mathbb{Z}^+$ such that $d(x, x_i) < 1$. Suppose that $i_i, \ldots, i_\ell \in \mathbb{Z}^+$ have been chosen for some $\ell \in \mathbb{Z}^+$. Using (Δ), we can find $i_{\ell+1} \in \mathbb{Z}^+$ such that $i_{\ell+1} > i_\ell$ and $d(x, x_{\ell+1}) < \dfrac{1}{\ell+1}$. Continuing in this way, we get a subsequence $\{x_{i_\ell}\}_{\ell=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that $\{x_{i_\ell}\}_{\ell=1}^{\infty}$ converges to x, as desired.

Suppose now that K is sequentially compact. Let L be an arbitrary infinite subset of K. Because L has infinitely many elements, we can find an infinite sequence $\{x_i\}_{i=1}^{\infty}$ of distinct elements of L. Since K is sequentially compact, there is a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ and an element $x \in K$ such that $\{x_{i_n}\}_{n=1}^{\infty}$ converges to x in $\mathcal{M}$. We will show that x is a limit point of L. Let $r > 0$ be given. Because $\{x_{i_n}\}_{n=1}^{\infty}$ converges to x in $\mathcal{M}$, there is an $N \in \mathbb{Z}^+$ such that $d(x, x_{i_n}) < r$ for every $n \geq N$. This implies that there are infinitely many elements of L in $B(x, r)$ because the x_i's are distinct elements of L. In particular, there is an element of L different from x in $B(x, r)$, as desired. □

§20 October 16, 2019

§20.1 Cauchy Sequences

Definition 20.1. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ is a **Cauchy sequence** if for every $\epsilon > 0$, there is an $L \in \mathbb{Z}^+$ such that for every $i, j \in \mathbb{Z}^+$ with $i, j \geq L$, we have that $d(x_i, x_j) < \epsilon$.

Proposition 20.2 (Convergent Sequences Are Cauchy)
If $\{x_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathcal{M}$ converges to an element x of $\mathcal{M}$, then $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}$.

Proof. Let $\epsilon > 0$ be given. Since $\{x_i\}_{i=1}^{\infty}$ converges to $x \in \mathcal{M}$, there is an $L \in \mathbb{Z}^+$ such that $d(x, x_i) < \frac{\epsilon}{2}$ for every $i \geq L$. If $i, j \geq L$, then $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, as desired. □

Proposition 20.3 (Cauchy Sequences Are Bounded)
If $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}$, then $\{x_i\}_{i=1}^{\infty}$ is bounded in $\mathcal{M}$.

Proof. If $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence, then there is an $L \in \mathbb{Z}^+$ such that $d(x_i, x_j) < 1$ for every $i, j \geq L$. Let $R = \max_{1 \leq i \leq L-1} d(x_i, x_L)$ if $L \geq 2$ and 0 if $L = 1$. Using this, we get that $d(x_i, x_L) < R + 1$ for every $i \in \mathbb{Z}^+$, or equivalently, $x_i \in B(x_L, R+1)$ for every $i \in \mathbb{Z}^+$, as desired. □

Proposition 20.4 (Cauchy Sequences Converge to Limit of Subsequence)
If a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ of a Cauchy sequence $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{M}$, then $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{M}$.

Proof. Let $\epsilon > 0$ be given. Since $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence, there is an $L \in \mathbb{Z}^+$ such that $d(x_i, x_j) < \frac{\epsilon}{2}$ for every $i, j \geq L$. Because $\{x_{i_n}\}_{n=1}^{\infty}$ converges to x in $\mathcal{M}$, there is an $N \in \mathbb{Z}^+$ such that $d(x, x_{i_n}) < \frac{\epsilon}{2}$ for every $n \geq N$. If $i_n, j \geq L$ and $n \geq N$, then we get that $d(x, x_j) \leq d(x, x_{i_n}) + d(x_{i_n}, x_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (Δ). Note that $i_n \geq n$ for each $n \in \mathbb{Z}^+$ since $\{i_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers. If $n \geq \max(L, N)$, then $n \geq N$ and $i_n \geq n \geq L$. It follows that $d(x, x_j) < \epsilon$ for every $j \geq L$, by applying (Δ) with $n \geq \max(L, N)$, as desired. □

§21 October 18, 2019

Corollary 21.1
Let $\{x_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $\mathcal{M}$ and K be a sequentially compact subset of $\mathcal{M}$. If $x_i \in K$ for each $i \in \mathbb{Z}^+$, then $\{x_i\}_{i=1}^{\infty}$ converges to an element x of K in $\mathcal{M}$.

§21.1 Complete Metric Spaces

Definition 21.2. A metric space $(\mathcal{M}, d(x, y))$ is **complete** if every Cauchy sequence of elements of $\mathcal{M}$ converges to an element of $\mathcal{M}$.

Proposition 21.3 (Metric Space of Reals Is Complete)
$\mathbb{R}^n$ is complete with respect to the standard Euclidean metric.

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary Cauchy sequence of elements of $\mathbb{R}^n$. Note that this sequence is bounded by Proposition 20.3. Then it is easy to see that there is a cell C in $\mathbb{R}^n$ such that $x_i \in C$ for each $i \in \mathbb{Z}^+$. By Theorem 9.6, C is compact, so it has the limit point property. Thus, C is sequentially compact by Proposition 19.8. By Corollary 21.1, we get that $\{x_i\}_{i=1}^{\infty}$ converges to an element of C in $\mathbb{R}^n$, as desired. □

§21.2 Some Particular Sequences

1. $\lim\limits_{j\to\infty} \dfrac{1}{j^p} = 0$ for $p \in \mathbb{R}^+$.

   Proof. Let $\epsilon > 0$ be given. Let L be a positive integer such that $L > \left(\dfrac{1}{\epsilon}\right)^{\frac{1}{p}}$ (using the Archimedean property). If $j \geq L$, then $\dfrac{1}{j^p} < \epsilon$, as desired. □

2. $\lim\limits_{n\to\infty} r^n = 0$ for positive real $r < 1$.

   Proof. Let $a = \dfrac{1}{r} - 1 > 0$ and note that $\dfrac{1}{r^n} = (1 + a)^n \geq 1 + na$ for every $n \in \mathbb{Z}^+$. Thus, $r^n \leq \dfrac{1}{1 + na}$ for every $n \in \mathbb{Z}^+$. If $\alpha \in \mathbb{R}$ and $\alpha < 1$, then $n^\alpha r^n \leq \dfrac{n^\alpha}{1 + an} \leq \left(\dfrac{1}{a}\right) n^{\alpha-1}$ for every $n \in \mathbb{Z}^+$. This implies that $\lim\limits_{n\to\infty} n^\alpha r^n = 0$ because $\lim\limits_{n\to\infty} n^{\alpha-1} = 0$. as in (1).

   If β is any real number, then let K be a positive integer such that $\beta < k$ (using the Archimedean property). Note that $\dfrac{\beta}{k} < 1$ and $n^{\frac{1}{k}} < 1$, so that $\lim\limits_{n\to\infty} n^{\frac{\beta}{k}} (r^{\frac{1}{k}})^n = 0$. It follows that $n^\beta r^n = \left(n^{\frac{\beta}{k}} (r^{\frac{1}{k}})^n\right)^k \to 0$ as $n \to \infty$.

   In particular, if $z \in \mathbb{C}$ and $|z| < 1$, then $|z^n| = |z|^n \to 0$ as $n \to \infty$ so that $\lim\limits_{n\to\infty} z^n = 0$ in $\mathbb{C}$. □

3. $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = 1$ for $p \in \mathbb{R}^+$.

   Proof. Suppose that $p \geq 1$ so that $p^{\frac{1}{n}} \geq 1$ for every $n \in \mathbb{Z}^+$. Let $\epsilon > 0$ be given. We will show that $\left|p^{\frac{1}{n}} - 1\right| = p^{\frac{1}{n}} - 1 < \epsilon$ for sufficiently large n. This is equivalent to saying that for sufficiently large n, $p^{\frac{1}{n}} < 1 + \epsilon$, which is equivalent to $p < (1 + \epsilon)^n$. This is also the same as saying that $\dfrac{p}{(1 + \epsilon)^n} < 1$ for sufficiently large n, which holds because $\lim\limits_{n\to\infty} \dfrac{p}{(1 + \epsilon)^n} = 0$, as in (2).

If $0 < p < 1$, then we can apply the previous statement to $\frac{1}{p}$ to get that $\lim_{n\to\infty} (\frac{1}{p})^{\frac{1}{n}} < 1$. It follows that $p^{\frac{1}{n}} = \frac{1}{(\frac{1}{p})^{\frac{1}{n}}} \to 1$ as $n \to \infty$. ☐

4. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

   Proof. Let $\epsilon > 0$ be given. We show that for all sufficiently large $n \in \mathbb{Z}^+$, we have $\left|n^{\frac{1}{n}} - 1\right| = n^{\frac{1}{n}} - 1 < \epsilon$, which is equivalent to saying $n^{\frac{1}{n}} < 1 + \epsilon$. This is equivalent to saying $n < (1+\epsilon)^n$ for all sufficiently large $n \in \mathbb{Z}^+$, which is the same as $\frac{n}{(1+\epsilon)^n} < 1$. This holds because $\frac{n}{(1+\epsilon)^n} = n\left(\frac{1}{1+\epsilon}\right)^n \to 0$ as $n \to \infty$. ☐

§22 October 21, 2019

§22.1 Infinite Series

Definition 22.1. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of complex numbers. The **infinite series** $\sum_{i=1}^{\infty} a_i$ **converges** if the corresponding sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ converges in $\mathbb{C}$ with respect to the standard Euclidean metric. Then $\sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$.

Proposition 22.2 (Sums and Multiples of Convergent Series)

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be convergent series of complex numbers. Then $\sum_{i=1}^{\infty}(a_i + b_i)$ converges to $\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} ca_i$ converges to $c\sum_{i=1}^{\infty} a_i$ for $c \in \mathbb{C}$.

Proof. Observe that $\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i$. Then we can use the results from previous propositions about convergent sequences. ☐

Proposition 22.3 (The Cauchy Criterion)

An infinite series $\sum_{i=1}^{\infty} a_i$ of complex numbers converges if and only if for every $\epsilon > 0$, there is an $L \in \mathbb{Z}^+$ such that $\left|\sum_{i=j}^{n} a_i\right| < \epsilon$ for every $L \le j \le n$.

Proof. Let $(*)$ be the latter criterion. It is easy to see that $(*)$ is the same as saying that the sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ is a Cauchy sequence in $\mathbb{C}$. By Proposition 20.2, the conclusion follows. ☐

Proposition 22.4 (Real Series Converges $\iff$ Partial Sums Are Bounded)

An infinite series $\sum_{i=1}^{\infty} a_i$ of nonnegative real numbers converges if and only if the sequence of partial sums $s_n = \sum_{i=n}^{\infty} a_i$ is bounded.

Proof. The only-if part works for any convergent series of complex numbers by Proposition 17.16. Conversely, if a_i is a nonnegative real number for each $i \in \mathbb{Z}^+$, then it is easy to see that the sequence of partial sums is monotonically increasing in $\mathbb{R}$. Thus, the if part follows by Proposition 19.2. □

§22.2 Absolute Convergence

Definition 22.5. An infinite series $\sum_{i=1}^{\infty} a_i$ of complex numbers **converges absolutely** if $\sum_{i=1}^{\infty} |a_i|$ converges in $\mathbb{R}$.

Proposition 22.6 (Absolute Convergence Implies Convergence)

If an infinite series $\sum_{i=1}^{\infty} a_i$ converges absolutely, then $\sum_{i=1}^{\infty} a_i$ also converges.

Proof. Note that $\left|\sum_{i=j}^{n} a_i\right| \leq \sum_{i=j}^{n} |a_i|$ for each $1 \leq j \leq n$ by the triangle inequality. By Proposition 22.3, the conclusion follows. □

Proposition 22.7 (The Comparison Test)

Let $\sum_{i=1}^{\infty} a_i$ be an infinite series of complex numbers and $\sum_{i=1}^{\infty} b_i$ be an infinite series of nonnegative real numbers. If $|a_i| \leq b_i$ for each $i \in \mathbb{Z}^+$ and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges absolutely.

Proof. Note that $\sum_{i=1}^{\infty} |a_i| \leq \sum_{i=1}^{\infty} b_i$ for each $n \in \mathbb{Z}^+$ $(*)$. If $\sum_{i=1}^{\infty} b_i$ converges, then the corresponding sequence of partial sums $\sum_{i=1}^{\infty} b_i$ is bounded by Proposition 22.4. It follows that $\sum_{i=1}^{\infty} |a_i|$ converges, as desired. □

§23 October 23, 2019

§23.1 Geometric Series

Remark 23.1. Let $z \in \mathbb{C}$ be given and let $n \in \mathbb{Z}_{\geq 0}$. Then

$$(1-z)\sum_{i=0}^{n} z^i = 1 - z^{n+1}$$

where $z^0 = 1$. Indeed, $(1-z)\sum_{i=0}^{n} z^i = \sum_{i=0}^{n} z^i - \sum_{i=0}^{n} z^{i+1} = \sum_{i=0}^{n} z^i - \sum_{i=1}^{n+1} z^i = 1 - z^{n+1}$.

Remark 23.2. If $|z| < 1$, then $|z^n| = |z|^n \to 0$ as $n \to \infty$, so that $\lim_{n\to\infty} z^n = 0$.

Remark 23.3. If $z \neq 1$, then $\sum_{i=0}^{n} z^i = \frac{1-z^{n+1}}{1-z}$ for every $n \geq 0$.

Remark 23.4. If $|z| < 1$, then $\sum_{i=0}^{n} z^i$ converges with $\sum_{i=0}^{\infty} z^i = \frac{1}{1-z}$.

Remark 23.5. If $\sum_{i=1}^{\infty} a_i$ is a convergent series of complex numbers, then $\lim_{\ell\to\infty} a_\ell = 0$. This follows from the Cauchy criterion.

Remark 23.6. If $z \in \mathbb{C}$ and $|z| \geq 1$, then $\left|z^i\right| = |z|^i \geq 1$ for every $i \geq 0$. In particular, $\left|z^i\right|_{i=1}^{\infty}$ does not converge to 0 in $\mathbb{C}$, and hence $\sum_{i=1}^{\infty} z^i$ does not converge.

§23.2 Power Series

Definition 23.7. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of complex numbers and consider the corresponding series $\sum_{j=0}^{\infty} a_j z^j$ where $z \in \mathbb{C}$.

Proposition 23.8 (Smaller Absolute Value Series Converges)

If $\sum_{j=0}^{\infty} a_j z^j$ converges for some $z \in \mathbb{C}$ and $w \in \mathbb{C}$ satisfies $|w| < |z|$, then $\sum_{j=0}^{\infty} a_j w^j$ converges absolutely

Proof. Since $\sum_{j=0}^{\infty} a_j z^j$ converges, then $\{a_j z^j\}_{j=0}^{\infty}$ converges to 0 in $\mathbb{C}$, and hence this set is bounded in $\mathbb{C}$. This means that there is a nonnegative real number C such that $\left|a_j z^j\right| \leq C$ for every $j \geq 0$. It follows that $\left|a_j w^j\right| = \left|a_j z^j\right| \left|\frac{w^j}{z^j}\right| = C\left(\frac{|w|}{|z|}\right)^j$ for each $j \geq 0$. If $|w| < |z|$, then $\frac{|w|}{|z|} < 1$, so that $\sum_{j=0}^{\infty} \left(\frac{|w|}{|z|}\right)^j$ is a convergent geometric series. This implies that $\sum_{j=0}^{\infty} C\left(\frac{|w|}{|z|}\right)^j$ converges. It follows that $\sum_{j=0}^{\infty} \left|a_j w^j\right|$ converges by the comparison test, as desired. □

Remark 23.9. If $\sum_{j=0}^{\infty} a_j z^j$ converges absolutely for some $z \in \mathbb{C}$

Definition 23.10. Let E be the set of $z \in \mathbb{C}$ such that $\sum_{j=0}^{\infty} a_j z^j$ does not converge, and let $E_1 = \{|z| \mid z \in E\}$. The **radius of convergence** ρ of $\sum_{j=0}^{\infty} a_j z^j$ is defined by $\rho = \inf E_1$ when $E \neq \emptyset$ and $\rho = +\infty$ when $E = \emptyset$.

Proposition 23.11

ρ is uniquely determined by the following conditions:

1. If $w \in \mathbb{C}$ and $|w| < \rho$, then $\sum_{j=0}^{\infty} a_j w^j$ converges absolutely. (This holds for all $w \in \mathbb{C}$, when $\rho = +\infty$.
2. If $w \in \mathbb{C}$ and $|w| > \rho$, then $\sum_{j=0}^{\infty} a_j w^j$ does not converge. (This is vacuous when $\rho = +\infty$).

Proof. We prove (1). Suppose that $w \in \mathbb{C}$ and $|w| < \rho$. It is easy to see that there is a $z \in \mathbb{C}$ such that $|w| < |z| < \rho$. take $z = \frac{|w| + \rho}{2}$ when $\rho < +\infty$ and $z = |w| + 1$ when $\rho = +\infty$. Note that $|z| \neq E$, by the definition of ρ. This implies that $z \notin E$, so $\sum_{j=0}^{\infty} a_j z^j$ converges by definition of E. By Proposition 23.8, $\sum_{j=0}^{\infty} a_j w^j$ converges absolutely, as desired.

Now we show (2). Suppose that $w \in \mathbb{C}$ and $\sum_{j=0}^{\infty} a_j w^j$ converges. We show that $|w| \leq \rho$. If $z \in \mathbb{C}$ and $|z| < |w|$, then $\sum_{j=0}^{\infty} a_j z^j$ converges absolutely by the previous proposition. This means that $z \notin E$, so that $|w|$ is a lower bound for E_1 It follows that $|w| \leq \rho$ by definition of ρ, as desired.

Now we prove the proposition. □

§24 October 25, 2019

Proposition 24.1 (The Cauchy Condensation Test)

Let $a_1, a_2, \ldots$ be a monotonically decreasing sequence of nonnegative real numbers. Then $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof. Let k be a nonnegative integer. Then $\displaystyle\sum_{i=2^k}^{2^{k+1}-1} a_i \leq \sum_{i=2^k}^{2^{k+1}-1} a_{2^k} = 2^k a_2$. Let m, n be positive integers with $m \leq 2^{n+1} - 1$. Note that

$$\sum_{i=1}^{m} a_i \leq \sum_{k=0}^{n} \left(\sum_{i=2^k}^{2^{k+1}-1} a_i \right) \leq \sum_{k=0}^{n} 2^k a_{2^k}.$$

If $\displaystyle\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then the partial sums $\displaystyle\sum_{k=0}^{n} 2^k a_{2^k}$ are bounded, by Proposition 22.4. This implies that the partial sums $\displaystyle\sum_{i=1}^{m} a_i$ are bounded by the previous estimation. It follows that $\displaystyle\sum_{i=1}^{\infty} a_i$ converges, as desired.

Now we prove the other direction. If $k \in \mathbb{Z}^+$, then $2^k a_{2^k} = 2 \cdot 2^{k-1} a_{2^k} \leq 2 \displaystyle\sum_{i=2^{k-1}+1}^{2^k} a_i$. If $n \in \mathbb{Z}^+$, then $\displaystyle\sum_{k=0}^{n} 2^k a_{2^k} = a_1 + \sum_{k=1}^{n} 2^k a_{2^k} \leq a_1 + 2 \sum_{k=1}^{n} \left(\sum_{i=2^{k-1}+1}^{2^k} a_i \right) \leq= a_1 + 2 \sum_{i=1}^{2^n} a_i \leq 2 \sum_{i=1}^{2^n} a_i$. If $\displaystyle\sum_{i=1}^{\infty} a_i$ converges, then the partial sums $\displaystyle\sum_{i=1}^{\infty} a_i$ are bounded. This implies that the partial sums $\displaystyle\sum_{k=0}^{n} 2^k a_{2^k}$ are bounded, so that $\displaystyle\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, as desired. □

Example

Let p be a nonnegative real number, and consider $a_i = \dfrac{1}{i^p}, i \in \mathbb{Z}$. Then $2^k a_{2^k} = 2^k \left(\dfrac{1}{(2^k)^p} \right) = 2^{(1-p)k}$ for each $k \geq 0$. Note that $\displaystyle\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges if and only if $p > 1$ since this is a geometric sequence. Thus, it follows that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges if and only if $p > 1$.

Theorem 24.2
Let $\{a_j\}_{j=0}^{\infty}$ be a sequence of complex numbers such that the corresponding sequence of partial sums $A_n = \displaystyle\sum_{j=0}^{n} a_j$ is bounded, and let $\{b_j\}_{j=0}^{\infty}$ be a monotonically decreasing sequence of nonnegative real numbers that converges to 0 in $\mathbb{R}$. Then $\displaystyle\sum_{j=0}^{\infty} a_j b_j$ converges.

Proof. Let $A_{-1} = 0$, so that $a_j = A_j - A_{j-1}$ for each $j \geq 0$. If n is a nonnegative integer, then $\displaystyle\sum_{j=0}^{n} A_j b_j - \sum_{j=0}^{n} A_{i-1} b_i$. We also have that $\displaystyle\sum_{j=0}^{n} A_{j-1} b_j = \sum_{j=-1}^{n-1} A_j b_{j+1} =$

$\sum_{j=0}^{n} A_j b_{j+1} + A_{-1}b_0 - A_n b_{n+1}$. It follows that $\sum_{j=0}^{n} a_j b_j = \sum_{j=0}^{n} A_j b_j - \sum_{j=0}^{n} A_j b_{j+1} + A_n b_{n+1} = \sum_{j=0}^{n} A_j(b_j - b_{j+1}) + A_n b_{n+1}$. Observe that $\lim_{n\to\infty} A_n b_{n+1} = 0$ because $\{A_n\}_{n=0}^{\infty}$ is bounded and $\{b_{n+1}\}_{n=1}^{\infty}$ converges to 0. It suffices to show that $\sum_{j=0}^{\infty} A_j(b_j - b_{j+1})$ converges.

Since $\{A_j\}_{j=0}^{\infty}$ is bounded, there is a nonnegative real number C such that $|A_j| \leq C$ for every $j \geq 0$. Thus, $|A_j(b)j - b_{j+1}| = |A_j|\,|b_j - b_{j+1}| \leq C(b_j - b_{j+1})$. It suffices to show that $\sum_{j=0}^{\infty} C(b_j - b_{j+1}$ converges, by the comparison test. Of course, it is enough to show that $\sum_{j=0}^{\infty}(b_j - b_{j+1})$ converges. Note that for each nonnegative integer n, $\sum_{j=0}^{n}(b_j - b_{j+1}) = \sum_{j=0}^{n} b_j - \sum_{j=0}^{n} b_{j+1} = \sum_{j=0}^{n} b_j - \sum j = 1^{n+1} b_j = b_0 - b_{n+1}$. Hence, $\lim_{n\to\infty}\sum_{j=0}^{n}(b_j - b_{j+1}) = b_0$, because $\lim_{n\to\infty} b_{n+1} = 0$ by assumption. Thus, $\sum_{j=0}^{\infty}(b_j - b_{j+1})$ converges, as desired. □

Example

Let $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq 1$. Let $a_j = z^i$ for each $j \geq 0$. Then $A_n = \sum_{j=0}^{n} z^i = \frac{1 - z^{n+1}}{1 - z}$ for every $n \geq 0$. In this case, $|A_n| = \frac{|1 - z^{n+1}|}{|1 - z|} \leq \frac{|1| + |z^{n+1}|}{|1 - z|} = \frac{1 + |z|^{n+1}}{|1 - z|} = \frac{2}{|1 - z|}$. Thus $\{A_n\}_{n=0}^{\infty}$ is bounded. If $\{b_j\}_{j=0}^{\infty}$ is as in the theorem, then we get that $\sum_{j=0}^{\infty} z^j b_j$ converges.

In particular, we can take $z = -1$, which gives Leibniz's **alternating series test**.

§25 October 28, 2019

Definition 25.1. Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be infinite series of complex numbers. Let $c_n = \sum_{j=0}^{n} a_j b_{n-j}$ for each nonnegative integer n. The infinite series $\sum_{n=0}^{\infty} c_n$ is called the **Cauchy product** of $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$.

Remark 25.2. It is easy to see that $\sum_{n=0}^{\infty} c_n = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{j=0}^{\infty} b_j\right)$. In particular, if $a_j = 0$ for all but finitely many $j \geq 0$, and if $b_\ell = 0$ for all but finitely many $\ell \geq 0$, then it is easy to see that $c_n = 0$ for all but finitely many $n \geq 0$ and that the above identity holds.

Remark 25.3. Similarly, if $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ are power series in z with complex coefficients, then $\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{j=0}^{\infty} b_j z^j\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty} a_j z^j b_{n-j} z^{n-j}\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty} a_j b_{n-j}\right) z^n = \sum_{n=0}^{\infty} c_n z^n$.

Remark 25.4. Suppose that $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ are convergent series of nonnegative real numbers. If N is a nonnegative integer, then $\sum_{n=0}^{N} c_n = \sum_{n=0}^{N}\left(\sum_{j=0}^{n} a_j b_{n-j}\right) \leq \left(\sum_{j=0}^{N} a_j\right)\left(\sum_{j=0}^{N} b_j\right) \leq \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{j=0}^{\infty} b_j\right)$. This implies that $\sum_{n=0}^{\infty} c_n$ converges with $\sum_{n=0}^{\infty} c_n \leq \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{j=0}^{\infty} b_j\right)$. Similarly, if N_1 and N_2 are nonnegative integers, then $\left(\sum_{j=0}^{N_1} a_j\right)\left(\sum_{j=0}^{N_2} b_j\right) \leq \sum_{n=0}^{N_1+N_2} c_n \leq \sum_{n=0}^{\infty} c_n$. It follows that $\left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{j=0}^{\infty} b_j\right) \leq \sum_{n=0}^{\infty} c_n$, by taking $N_1 = N_2 \to \infty$.

Remark 25.5. Suppose now that $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ are absolutely convergent series of complex numbers. Observe that $|c_n| = \left|\sum_{j=0}^{n} a_j b_{n-j}\right| \leq \sum_{j=0}^{n} |a_j|\,|b_{n-j}|$, which is the nth term of the Cauchy product of $\sum_{j=0}^{\infty} |a_j|$ and $\sum_{j=0}^{\infty} |b_j|$. By the previous statements applied to these series and the comparison test, we get that $\sum_{n=0}^{\infty} |c_n|$ converges, with $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} |a_j|\,|b_{n-j}|\right) = \left(\sum_{j=0}^{\infty} |a_j|\right)\left(\sum_{j=0}^{\infty} |b_j|\right)$. In particular, this means that $\sum_{n=0}^{\infty} c_n$ converges absolutely. More precisely, this can be reduced to the corresponding statement for convergent series of nonnegative real numbers, by expressing $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ as linear combinations of convergent series of nonnegative real numbers.

Remark 25.6. If $\sum_{j=0}^{\infty} a_j$ converges absolutely and $\sum_{j=0}^{\infty} b_j$ converges, then it is easy to see that $\sum_{n=0}^{\infty} c_n$ converges.

§26 October 30, 2019

§26.1 Extended Reals

Definition 26.1. The set of **extended real numbers** consists of the real numbers and two additional elements, denoted as $+\infty$ and $-\infty$. The standard ordering on the real line can be extended to the set of extended real numbers by putting $-\infty < x < +\infty$ for every $x \in \mathbb{R}$.

Remark 26.2. The notions of upper and lower bounds and infimum and supremum can also be extended to the set of extended real numbers in a simple way.

Let A be a nonempty set of extended real numbers. One can check that the supremum and infimum of A are always defined as extended real numbers. More precisely, the supremum of A can be categorized as follows.

1. If $+\infty \in A$, then $\sup A = +\infty$.
2. If $+\infty \notin A$, $A \cap \mathbb{R} \neq \emptyset$ and $A \cap \mathbb{R}$ has no upper bound in $\mathbb{R}$, then $\sup A = +\infty$.
3. If $+\infty \neq A$, $A \cap \mathbb{R} \neq \emptyset$, and $A \cap \mathbb{R}$ has an upper bound in $\mathbb{R}$, then $\sup A$ is the same as $\sup(A \cap \mathbb{R}) \in \mathbb{R}$.
4. If $A = \{-\infty\}$, then $\sup A = -\infty$.

The infimum can be characterized similarly. One can check that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$ in the set of extended real numbers.

Remark 26.3. Addition and subtraction are sometimes defined for extended real numbers, as follows. We put $+\infty + (+\infty) = +\infty$, $-\infty + (-\infty) = -\infty$, but the sum of $+\infty$ and $-\infty$ is not defined. If $x \in \mathbb{R}$, then $x + (+\infty) = +\infty + x = +\infty$ and $x + (-\infty) = -\infty + x = -\infty$.

If x is an extended real number and $x > 0$, then $x \cdot (+\infty) = (+\infty) \cdot x = +\infty$ and $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$. Similarly, if $x < 0$, then $x \cdot (+\infty) = (+\infty) \cdot x = -\infty$ and $x \cdot (-\infty) = (-\infty) \cdot x = +\infty$

If $x \in \mathbb{R}$, then $\dfrac{x}{+\infty} = \dfrac{x}{-\infty} = 0$. However, $\dfrac{+\infty}{-\infty}$ is not defined, nor is $\dfrac{1}{0}$. (Sometimes, it may be appropriate to interpret $\dfrac{1}{0}$ as being $+\infty$ if the extended real numbers under consideration are all nonnegative.) Similarly, $0 \cdot (\pm\infty)$ is not defined.

Definition 26.4. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers. We say that x_j **tends to** $+\infty$ as $j \to \infty$, or $x_j \to +\infty$ as $j \to \infty$, if for every $R \in \mathbb{R}$, there is an $L \in \mathbb{Z}^+$ such that if $j \geq L, j \in \mathbb{Z}^+$, then $x_j > R$. Similarly, x_j **tends to** $-\infty$ as $j \to \infty$ if for every $R \geq 0 \in \mathbb{R}$, there is an $L \in \mathbb{Z}^+$ such that for every $j \in \mathbb{Z}^+$ with $j \geq L$, we have that $x_j < -R$.

Proposition 26.5 (Properties of Sequences That Converge to Extended Reals)
Let $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ be sequences of real numbers, and suppose that there are extended real numbers x and y such that $x_j \to x$ and $y_j \to y$ as $j \to \infty$.

1. if $x+y$ is defined as an extended real number, then $x_j + y_j \to x + y$ as $j \to \infty$.
2. if xy is defined as an extended real number, then $x_j y_j \to xy$ as $j \to \infty$.
3. if $x_j \neq 0$ for each $j \in \mathbb{Z}^+$ and $x \neq 0$, then $\frac{1}{x_j} \to \frac{1}{x}$

Proof. If $x_j \to +\infty$ as $j \to \infty$ and the y_j's have a lower bound in $\mathbb{R}$, then one can check that $x_j + y_j \to +\infty$ as $j \to \infty$. In particular, if $y_j \to y$ as $j \to \infty$ and $y > -\infty$, then we can use this to get that $x_j + y_j \to +\infty$ as $j \to \infty$. We can use this to get part 1 of the proposition.

Suppose now that $x_j \to +\infty$ as $j \to \infty$, and that there is a positive real number a such that $y_j \geq a$ for all but finitely many $j \in \mathbb{Z}^+$. We can verify that $x_j y_j \to +\infty$ as $j \to \infty$. In particular, this holds when $y_j \to y$ as $j \to \infty$ and $y > 0$, with $a = \frac{y}{2}$. We can use this to get part 2 of the proposition.

If $|x_j| \to +\infty$ as $j \to \infty$ and $x_j \neq 0$ for each $j \in \mathbb{Z}+$, then it is easy to see that $\frac{1}{x_j} \to 0$ as $j \to \infty$. We can use this to get part 3 of the proposition. □

Remark 26.6. If $\{x_j\}_{j=1}^{\infty}$ is a sequence of positive real numbers that converges to 0 in $\mathbb{R}$, then it is easy to see that $\frac{1}{x_j} \to +\infty$ as $j \to \infty$.

§27 November 1, 2019

Proposition 27.1 (Any Real Sequence Has A Convergent Subsequence)
If $\{x_j\}_{j=1}^{\infty}$ is any sequence of real numbers, then there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ and an extended real number x such that $x_{j_\ell} \to x$ as $\ell \to \infty$.

Proof. We consider several cases.

Case 1: Suppose there are real numbers a and b such that $a \leq x_j \leq b$ for every $j \in \mathbb{Z}^+$. Then the proposition can be derived from the fact that $[a, b]$ is compact and hence sequentially compact.

Case 2: Suppose that for each $b \in \mathbb{R}$, there is a $j \in \mathbb{Z}^+$ such that $x_j > b$. This implies that for each $b_1 \in \mathbb{R}$, there are infinitely many $j \in \mathbb{Z}^+$ such that $x_j > b_1$. Otherwise, we would get a finite upper bound by taking the maximum of b_1 and the finitely many x_j's for which $x_j > b_1$. In particular, there is a $j_1 \in \mathbb{Z}^+$ such that $x_{j_1} > 1$. If $j_\ell \in \mathbb{Z}^+$ has already been chosen, then we can choose $j_{\ell+1} \in \mathbb{Z}^+$ such that $j_{\ell+1} > j_\ell$ and $x_{j_{\ell+1}} > \ell + 1$ by taking $b_1 = \ell + 1$. Continuing in this manner, we obtain a subsequence $\{x_{i_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $x_{j_\ell} > \ell$ for each $\ell \in \mathbb{Z}^+$. This implies that $x_{j_\ell} \to +\infty$ as $\ell \to \infty$, as desired.

Case 3: Suppose that for each $a \in \mathbb{R}$, we have that $x_j < a$ for some $j \in \mathbb{Z}^+$. This can be handled analogously as in Case 2 by considering $\{-x_j\}_{j=1}^{\infty}$. This concludes the proof. □

Definition 27.2. Let $\{x_j\}_{j=1}^{\infty}$ be any sequence of real numbers, and let E be the set of extended real numbers x for which there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $x_{j_\ell} \to x$ as $\ell \to \infty$. We define $\limsup_{j\to\infty} x_j = \sup E$ and $\liminf_{j\to\infty} x_j = \inf E$.

Remark 27.3. E is nonempty by Proposition 27.1.

Remark 27.4. $\sup E$ and $\inf E$ are extended real numbers.

Remark 27.5. $\liminf \leq \limsup$ since E is nonempty.

Example 27.6 (Examples of lim sup and lim inf)

Some examples:

1. If there is an extended real number x such that $x_j \to x$ as $j \to \infty$, then $E = \{x\}$ and hence $\limsup_{j\to\infty} x_j = \liminf_{j\to\infty} x_j = x$.
2. If $x_j = (-1)^j(1 + \frac{1}{j})$ for each $j \in \mathbb{Z}^+$, then we can check that $E = \{-1, 1\}$, so that $\limsup_{j\to\infty} x_j = 1$ and $\liminf_{j\to\infty} x_j = -1$.
3. If every rational number occurs in $\{x_j\}_{j=1}^{\infty}$, then we can check that E is the set of all extended real numbers. In particular, $\limsup_{j\to\infty} x_j = +\infty$ and $\limsup_{j\to\infty} x_j = -\infty$.

Proposition 27.7

Let $y = \limsup_{j\to\infty} x_j$.

1. If $z \in \mathbb{R}$, and $y < z$, then $x_i < z$ for all but finitely many $j \in \mathbb{Z}+$.
2. If $w \in \mathbb{R}$ and $w < y$, then $x_j > w$ for infinitely many $j \in \mathbb{Z}^+$.
3. y is uniquely determined by these two properties.

Proof. Proof of 1. Suppose for the sake of contradiction that $z \in \mathbb{R}$, $z > y$ and $x_j \geq z$ for infinitely many $j \in \mathbb{Z}^+$. This implies that there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $x_{j_\ell} \geq z$ for every $\ell \in \mathbb{Z}^+$. By Proposition 27.1, we get a subsequence $\{x_{j_{\ell_n}}\}_{n=1}^{\infty}$ of $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ and an extended real number x such that $x_{j_{\ell_n}} \to x$ as $n \to \infty$. Note that $\{x_{j_{\ell_n}}\}_{n=1}^{\infty}$ is also a subsequence of $\{x_j\}_{j=1}^{\infty}$, so that $x \in E$. By construction, $x_{j_{\ell_n}} \geq z$ for every $n \in \mathbb{Z}^+$, so we can check that $x \geq z$. This means that $x > y$. However, $x \leq y$ since $x \in E$ and $y = \sup E$. This is a contradiction, as desired.

Proof of 2. If $w \in \mathbb{R}$ and $w < y$, then there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$, and an extended real number x such that $x_{j_\ell} \to x$ as $\ell \to \infty$ and $x > w$. Otherwise, w would be an upper bound for E less than y. This implies that $x_{j_\ell} > w$ for all but finitely many $\ell \in \mathbb{Z}^+$. It follows that $x_j > w$ for infinitely many $j \in \mathbb{Z}^+$, as desired.

Proof of 3. Let y' and y'' be extended real numbers that satisfy the conditions in 1 and 2. Suppose, for the sake of contradiction, that $y' \neq y''$. Without loss of generality, let $y' < y''$. Thus, there must be a real number v such that $y' < v < y''$. Using condition

1, for y' and $z = v$, we get that $x_j < v$ for all but finitely many $j \in \mathbb{Z}^+$. Similarly, we can apply condition 2 for y'' and $w = v$ to get that $x_j > v$ for infinitely many $j \in \mathbb{Z}^+$. This is a contradiction, as desired. □

Proposition 27.8

Let $t = \liminf_{j\to\infty} x_j$

1. If $r \in \mathbb{R}$ and $r < t$, then $x_j > r$ for all but finitely many $j \in \mathbb{Z}^+$.
2. If $u \in \mathbb{R}$ and $u > t$, then $x_j < u$ for infinitely many $j \in \mathbb{Z}^+$.
3. t is uniquely determined by these two properties.

Proof. Same as above. □

§28 November 4, 2019

Corollary 28.1

Suppose that $\limsup_{j\to\infty} x_j = \liminf_{j\to\infty} x_j = x$. Then $x_j \to x$ as $j \to \infty$.

Proof. This can be obtained using part 1 of each of the previous propositions. More precisely, if $x = +\infty$, then it suffices to use part 1 of the second proposition. Similarly, if $x = -\infty$, then it suffices to use part 1 of the first proposition. □

§29 November 6, 2019

Theorem 29.1 (Root Test)

Let $\sum_{j=1}^{\infty} a_j$ be an infinite series of complex numbers, and let $\alpha = \limsup_{j\to\infty} |a_j|^{\frac{1}{j}}$.

1. If $\alpha < 1$, then $\sum_{j=1}^{\infty} a_j$ converges absolutely.
2. If $\alpha > 1$, then $\sum_{j=1}^{\infty} a_j$ does not converge.

Proof. If $\alpha < 1$, then let β be a real number with $\alpha < \beta < 1$. By Proposition 27.7, $|\alpha_j|^{\frac{1}{j}} < \beta$ for all but finitely many $j \in \mathbb{Z}^+$. This means that $|a_j| < \beta^j$ for all but finitely many $j \in \mathbb{Z}^+$. Note that $\sum_{j=1}^{\infty} \beta^j$ is a convergent geometric series because $\beta < 1$. By the comparison test, we have that $\sum_{j=1}^{\infty} |a_j|$ converges, as desired.

Suppose now that $\alpha > 1$, and let γ be a real number with $1 \le \gamma < \alpha$. Then $|a_j|^{\frac{1}{j}} > \gamma$ for infinitely many $j \in \mathbb{Z}^+$ by Proposition 27.7. This implies that $|a_j| > \gamma^j$ for infinitely many $j \in \mathbb{Z}^+$. If we take $\gamma = 1$, then we get that $\{a_j\}_{j=1}^{\infty}$ does not converge to 0, so that $\sum_{j=1}^{\infty} a_j$ does not converge. If we take $\gamma > 1$, we get that $\{a_j\}_{j=1}^{\infty}$ is not even bounded in $\mathbb{C}$. □

Corollary 29.2 (Radius of Convergence)

The radius of convergence of $\sum_{j=1}^{\infty} a_j z^j$ is $\frac{1}{\alpha}$

Proof. Let $z \in \mathbb{C}$ be given, and consider $\sum_{j=1}^{\infty} a_j z^j$. Of course, this series converges absolutely when $z = 0$, and so we suppose that $z \neq 0$. Observe that $\left|a_j z^j\right|^{\frac{1}{j}} = |a_j|^{\frac{1}{j}} \left(|z|^j\right)^{\frac{1}{j}} = |a_j|^{\frac{1}{j}} |z|$ for every $j \in \mathbb{Z}^+$. We can check that $\limsup_{j\to\infty} \left|a_j z^j\right|^{\frac{1}{j}} = |z| \limsup_{j\to\infty} |a_j|^{\frac{1}{j}} = |z|\,\alpha$.

It follows from the root test that $\sum_{j=1}^{\infty} a_j z^j$ converges absolutely when $|z| < 1$, and that the series does not converge when $|z|\,\alpha > 1$. This implies that the radius of convergence of $\sum_{j=1}^{\infty} a_j z^j$ is equal to $\frac{1}{\alpha}$, or $+\infty$ when $\alpha = 0$. □

Example 29.3

Some examples

1. If $a_j = 1$ for each $j \in \mathbb{Z}^+$, then $\alpha = 1$, and $\sum_{j=1}^{\infty} a_j$ does not converge.

2. If $a_j = \frac{1}{j^2}$ for each $j \in \mathbb{Z}^+$, then $|a_j|^{\frac{1}{j}} = \left(\frac{1}{j^2}\right)^{\frac{1}{j}} = \frac{1}{(j^{\frac{1}{j}})^2} \to 1$ as $j \to \infty$ because $\lim_{j\to\infty} j^{\frac{1}{j}} = 1$. In this case, $\sum_{j=1}^{\infty} a_j$ converges.

Theorem 29.4 (Ratio Test)

Let $\sum_{j=1}^{\infty} a_j$ be an infinite series of nonzero complex numbers.

1. If $\limsup_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} < 1$, then $\sum_{j=1}^{\infty} a_j$ converges absolutely.

2. If $\liminf_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} > 1$, then $\sum_{j=1}^{\infty} a_j$ does not converge.

Proof. Let β be a real number such that $\limsup_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} < \beta < 1$. This implies that there is an $L \in \mathbb{Z}^+$ such that $\frac{|a_{j+1}|}{|a_j|} < \beta$ for every $j \geq L$. This means that $|a_{j+1}| < \beta |a_j|$ for each $j \geq L$. It follows that $|a_j| \leq \beta^{j-L} |a_j|$ for every $j \geq L$. Note that $\sum_{j=1}^{\infty} \beta^{j-L} |a_L| = |a_L| \sum_{j=0}^{\infty} \beta^i$ converges, because $\beta < 1$. Hence, $\sum_{j=1}^{\infty} |a_j|$ converges, by the comparison test. This implies that $\sum_{j=1}^{\infty} |a_j|$ converges, as desired.

We now prove the second part. Let γ be a real number such that $\liminf_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} > \gamma \geq 1$. This implies that there is an $L \in \mathbb{Z}^+$ such that $\frac{|a_{j+1}|}{|a_j|} > \gamma$ for every $j \geq L$. It follows that $|a_{j+1}| > \gamma |a_j|$ for every $j \geq L$, so that $|a_j| \geq \gamma^{j-L} |a_L|$ for every $j \geq L$. If $\gamma - 1$, then we get that $|a_j| \geq |a_j|$ for every $j \geq L$, and in particular, that $\{a_j\}_{j=1}^{\infty}$ does not converge. If we take $\gamma > 1$, then we get that $|a_j| \to +\infty$ as $j \to \infty$. ☐

Example 29.5

Examples

1. If $a_j = 1$ for every $j \in \mathbb{Z}^+$, then $\frac{|a_{j+1}|}{|a_j|} = 1$ for every $j \in \mathbb{Z}^+$.
2. If $a_j = \frac{1}{j^2}$ for every $j \in \mathbb{Z}^+$, then $\frac{|a_{j+1}|}{|a_j|} = \frac{j^2}{(j+1)^2} = \frac{1}{(1+\frac{1}{j})^2} \to 1$ as $j \to \infty$.

§29.1 Continuity

Definition 29.6. Let $(\mathcal{M}, d(x,y))$ and $(\mathcal{N}, \rho(u,v))$ be metric spaces. Let f be a function defined on $\mathcal{M}$ with values in $\mathcal{N}$, which may also be called a **mapping** from $\mathcal{M}$ into $\mathcal{N}$. This can be expressed by $f : \mathcal{M} \to \mathcal{N}$. We say that f is **continuous** at a point $x \in \mathcal{M}$ if for every positive real number ϵ, there is a positive real number δ such that for every $y \in \mathcal{M}$ with $d(x,y) < \delta$, we have that $\rho(f(x), f(y)) < \epsilon$.

§30 November 8, 2019

Proposition 30.1 (Continuous Mappings At A Point Preserve Convergence)

A mapping f from $\mathcal{M}$ to $\mathcal{N}$ is continuous at a point $x \in \mathcal{M}$ if and only if the following condition holds: if $\{x_j\}_{j=1}^{\infty}$ is a sequence of elements of $\mathcal{M}$ that converges to x, then $\{f(x_j)\}_{j=1}^{\infty}$ converges to $f(x)$ in $\mathcal{N}$.

Proof. We show the forward direction. Let $\epsilon > 0$ be given. Since f is continuous at x, there is a $\delta > 0$ such that for every $y \in \mathcal{M}$ with $d(x,y) < \delta$, we have that $\rho(f(x), f(y)) < \epsilon$. Since $\{x_j\}_{j=1}^{\infty}$ converges to x in $\mathcal{M}$, there is an $L \in \mathbb{Z}^+$ such that for every $j \geq L$, we have that $d(x, x_J) < \delta$. Combining these two statements, we get that for every $j \geq L$, $\rho(f(x), f(x_j)) < \epsilon$, as desired.

We now prove the other direction. Let $\epsilon > 0$ be given. Suppose, for the sake of contradiction, that for every $\delta > 0$, there is a $y(\delta) \in \mathcal{M}$ such that $d(x, y(\delta)) < \delta$ and $\rho(f(x), f(y(\delta))) \geq \epsilon$. Let us now choose, for each $j \in \mathbb{Z}^+$, a point $x_j \in \mathcal{M}$ satisfying the properties of $y(\delta)$ with $\delta - \frac{1}{j}$ ($x_j = y(\frac{1}{j})$). It is easy to see that $\{x_j\}_{j=1}^{\infty}$ converges to x in $\mathcal{M}$ ($d(x, x_j) < \frac{1}{j}$ for each $j \in \mathbb{Z}^+$). However, $\{f(x_j)\}_{j=1}^{\infty}$ does not converge to $f(x)$ in $\mathcal{N}$, because $\rho(f(x), f(x_j)) \geq \epsilon$ for each $i \in \mathbb{Z}^+$. This contradicts the condition, as desired. □

Definition 30.2. A mapping f from $\mathcal{M}$ into $\mathcal{N}$ is **continuous** if f is continuous at x for every $x \in \mathcal{M}$.

Proposition 30.3 (Function Operations Preserve Continuity in $\mathbb{C}$)
Let f and g be continuous complex-valued functions on $\mathcal{M}$, with respect to the standard Euclidean metric on $\mathbb{C}$, as the range of f and g. Then $f + g$ and $f \circ g$ are also continuous in $\mathcal{M}$. Moreover, if $f(x) \neq 0$ for every $x \in \mathcal{M}$, then $\dfrac{1}{f}$ is also continuous.

Proof. This follows from the characterization of continuity in terms of sequences and previous results about convergent sequences in $\mathbb{C}$. □

Proposition 30.4 (Continuity $\iff$ Open Sets Are Preserved Under Inverse)
A mapping f from $\mathcal{M}$ into $\mathcal{N}$ is continuous if and only if for every open set V in $\mathcal{N}$, we have that $f^{-1}(V) = \{x \in \mathcal{M} \mid f(x) \in V\}$ is an open set in $\mathcal{M}$.

Proof. Suppose that f is continuous. Let V be an arbitrary open set in $\mathcal{N}$, and let an arbitrary element x of $f^{-1}(V)$ be given. Thus, $f(x) \in V$, so there is an $\epsilon > 0$ such that $\{z \in \mathcal{N} \mid \rho(f(x), z) < \epsilon\} \subseteq V$ because V is an open set in $\mathcal{N}$. Since f is continuous at x, we get that there is a $\delta > 0$ such that for every $y \in \mathcal{M}$ with $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \epsilon$. It follows that for every $y \in \mathcal{M}$ with $d(x, y) < \delta$, we have that $f(y) \in V$, so that $y \in f^{-1}(V)$. This means that $\{y \in \mathcal{M} \mid d(x, y) < \delta\} \subseteq f^{-1}(V)$, as desired.

Now we show the other direction. Let an arbitrary element x of $\mathcal{M}$ be given. We will show that f is continuous at x. Let $\epsilon > 0$ be also given, and note that $V = \{z \in \mathcal{N} \mid \rho(f(x), z) < \epsilon\}$ is an open set in $\mathcal{N}$. By assumption, $f^{-1}(V)$ is an open set in $\mathcal{M}$. Of course, $f(x) \in V$ by construction, so that $x \in f^{-1}(V)$. Because $f^{-1}(V)$ is an open set in $\mathcal{M}$, there is a $\delta > 0$ such that $\{y \in \mathcal{M} \mid d(x, y) < \delta\} \subseteq f^{-1}(V)$. This means that for every $y \in \mathcal{M}$ with $d(x, y) < \delta$, we have that $y \in f^{-1}(V)$, so that $f(y) \in V$ and hence $\rho(f(x), f(y)) < \epsilon$, as desired. □

§31 November 11, 2019

Corollary 31.1

A mapping f from $\mathcal{M}$ into $\mathcal{N}$ is continuous if and only if $f^{-1}(E)$ is a closed set in $\mathcal{M}$ for every closed subset E of $\mathcal{N}$.

Proof. It is easy to see that $f^{-1}(\mathcal{N} \setminus E) = \mathcal{M} \setminus f^{-1}(E)$ for every mapping f from $\mathcal{M}$ into $\mathcal{N}$ and every subset E of $\mathcal{N}$. The conclusion follows from Proposition 2.11. □

Let $(\mathcal{M}_j, d_j(x, y))$ for $j = 1, 2, 3$ be three metric spaces.

Proposition 31.2 (Transitivity of Continuity)

Let f be a continuous mapping from $\mathcal{M}_1$ into $\mathcal{M}_2$ and g be a continuous mapping from $\mathcal{M}_2$ into $\mathcal{M}_3$. Then the **composition** $g \circ f$ of f and g is continuous as a mapping from $\mathcal{M}_1$ into $\mathcal{M}_3$.

Proof by continuity. Let $x \in \mathcal{M}_1$ be given. We will show that $g \circ f$ is continuous at x. Since g is continuous at $f(x) \in \mathcal{M}_2$, there is an $\eta > 0$ such that for every $u \in \mathcal{M}_2$ with $d_2(f(x), u) < \eta$, we have that $d_3(g(f(x)), g(u)) < \eta$. Since f is continuous at x, there is a $\delta > 0$ such that for every $y \in \mathcal{M}_1$ with $d_1(x, y) < \delta$, we have that $d_2(f(x), f(y)) < \eta$. Thus, if $y \in \mathcal{M}_1$ satisfies $d_1(x, y) < \delta$, then we use the above statement with $u = f(y)$ to get that $d_3(g(f(x)), g(f(y))) < \epsilon$. □

Proof by sequences. Let $x \in \mathcal{M}_1$ be given. We will show that $g \circ f$ is continuous at x. Let $\{x_j\}_{j=1}^{\infty}$ be an arbitrary sequence of elements of $\mathcal{M}_1$ that converges to x. Since f is continuous at x, we have that $\{f(x_j)\}_{j=1}^{\infty}$ converges to $f(x)$ in $\mathcal{M}_2$ by Proposition 30.1. Similarly, since g is continuous at $f(x)$, we have that $\{g(f(x_j))\}_{j=1}^{\infty}$ converges to $g(f(x))$ in $\mathcal{M}_3$. This implies that $g \circ f$ is continuous at x by Proposition 30.1, as desired. □

Proof by open sets. Let W be an arbitrary open subset of $\mathcal{M}_3$. Because g is continuous, $g^{-1}(W)$ is an open set in $\mathcal{M}_2$ by Proposition 30.4. Similarly, since f is continuous, $f^{-1}(g^{-1}(W))$ is an open set in $\mathcal{M}$. We can check that $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$. Thus, $(g \circ f)^{-1}(W)$ is an open set in $\mathcal{M}_1$. This implies that $g \circ f$ is continuous, by Proposition 30.4. □

Theorem 31.3 (Continuity Preserves Compactness)

Let f be a continuous mapping from $\mathcal{M}$ into $\mathcal{N}$. If K is a compact subset of $\mathcal{M}$, then $f(K) = \{f(x) \mid x \in K\}$ is a compact subset of $\mathcal{N}$.

Proof. Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $f(K)$ in $\mathcal{N}$. Observe that $f^{-1}(V_\alpha)$ is an open set in $\mathcal{M}$ for every $\alpha \in A$ because f is continuous and V_α is an open set in $\mathcal{N}$. Of course, $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$ because $\{V_\alpha\}_{\alpha \in A}$ covers $f(K)$ by assumption. We can check that $K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$. Thus, $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is an open covering of K in $\mathcal{M}$. If K is compact in $\mathcal{M}$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{j=1}^{n} f^{-1}(V_{\alpha_j})$. We can verify that $f(n) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}$ and $f(f^{-1}(V_{\alpha_j})) \subseteq V_{\alpha_j}$ for each $j = 1, \ldots, n$. □

Proof by sequential compactness. Suppose that K is sequentially compact in $\mathcal{M}$. We will show that $f(K)$ is sequentially compact in $\mathcal{N}$. Let $\{z_j\}_{j=1}^{\infty}$ be an arbitrary sequence of elements of $f(K)$. For each $j \in \mathbb{Z}^+$, choose an element x_j of K such that $f(x_j) = z_j$, which is possible since $z_j \in f(K)$ by assumption. Thus, $\{x_j\}_{j=1}^{\infty}$ is a sequence of elements of K, so that the sequential compactness of K implies that there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ converges to an element x of K in $\mathcal{M}$. Using the same sequence $\{j_\ell\}_{\ell=1}^{\infty}$ of indices, we get a corresponding subsequence $\{z_{j_\ell}\}_{\ell=1}^{\infty} = \{f(x_{j_\ell})\}_{\ell=1}^{\infty}$ of $\{z_j\}_{j=1}^{\infty}$. By the continuity of f at x, we have that $\{f(x_{j_\ell})\}_{\ell=1}^{\infty}$ converges to $f(x)$ in $\mathcal{N}$, by Proposition 30.1. Thus, $\{z_{j_\ell}\}_{\ell=1}^{\infty}$ converges to $f(x) \in f(K)$ in $\mathcal{N}$, as desired. □

§32 November 13, 2019

Corollary 32.1 (Extreme Value Theorem)

Let f be a continuous real-valued function on $\mathcal{M}$ with the standard Euclidean metric on $\mathbb{R}$ as the range of f. If K is a nonempty compact subset of $\mathcal{M}$, then there are elements p, q of K such that $f(q) \leq f(x) \leq f(p)$ for every $x \in K$.

Proof. By the previous theorem, $f(K)$ is a compact subset of $\mathbb{R}$. This implies that $f(K)$ is bounded in $\mathbb{R}$. Note that $f(K) \neq \emptyset$ since $K \neq \emptyset$ by assumption. It follows that $f(K)$ has a supremum and infimum in $\mathbb{R}$. We also have that $f(K)$ is a closed set in $\mathbb{R}$ since $f(K)$ is compact, so that $f(K)$ contains its infimum and supremum, as desired. □

§32.1 Uniform Continuity

Definition 32.2. A mapping f from $\mathcal{M}$ into $\mathcal{N}$ is **uniformly continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $x, y \in \mathcal{M}$ with $d(x, y) < \delta$, we have that $\rho(f(x), f(y)) < \epsilon$.

Remark 32.3. Uniformly continuous functions are continuous.

Theorem 32.4 (Compactness and Continuity Implies Uniform Continuity)

If $\mathcal{M}$ is compact and f is a continuous mapping from $\mathcal{M}$ into $\mathcal{N}$, then f is uniformly continuous.

Proof. Let $\epsilon > 0$ be given. If $x \in \mathcal{M}$, then there is a $\delta(x) > 0$ such that for every $y \in \mathcal{M}$ with $d(x, y) < \delta(x)$, we have that $\rho(f(x), f(y)) < \frac{\epsilon}{2}$. Let $B(x) = \{y \in \mathcal{M} \mid d(x, y) < \frac{\delta(x)}{2}\}$ for each $x \in \mathcal{M}$. Thus, $B(x)$ is an open ball in $\mathcal{M}$ centered at x, and hence an open set in $\mathcal{M}$ that contains x. This implies that $\{B(x)\}_{x \in \mathcal{M}}$ is an open covering of $\mathcal{M}$. Since $\mathcal{M}$ is compact, there are finitely many elements $x_1, \ldots, x_\ell$ of $\mathcal{M}$ such that $\mathcal{M} \subseteq \bigcup_{j=1}^{\ell} B(x_j)$. Let $\delta = \min_{1 \leq j \leq \ell} \dfrac{\delta(x_j)}{2} > 0$. Let $x, y \in \mathcal{M}$ be given, with $d(x, y) < \delta$. Thus, there is a $j \in \{1, \ldots, \ell\}$ such that $x \in B(x_j)$. Thus, $d(x_j, x) < \frac{\delta(x_j)}{2}$. By the triangle inequality, it follows that $d(x_j, y) \leq d(x_j, x) + d(x, y) < \frac{\delta(x_j)}{2} + \delta \leq \frac{\delta(x_j)}{2} + \frac{\delta(x_j)}{2} = \delta(x_j)$. By the definition of $\delta(x_j)$, we get that $\rho(f(x_j), f(x)) < \frac{\epsilon}{2}$ and $\rho(f(x_j), f(y)) < \dfrac{\epsilon}{2}$. By the triangle inequality, we get that $\rho(f(x), f(y)) \leq \rho(f(x), f(x_j)) + \rho(f(x_j), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, as desired. □

§33 November 15, 2019

Proof 2. Let $\epsilon > 0$. Suppose, for the sake of contradiction, that for every $\delta > 0$, there are $x(\delta), y(\delta) \in \mathcal{M}$ such that $d(x(\delta), y(\delta)) < \delta$ and $\rho(f(x), f(y)) \geq \epsilon$. Thus, for each $j \in \mathbb{Z}^+$, we can choose $x_j, y_j \in \mathcal{M}$ such that $d(x_j, y_j) < \frac{1}{j}$ and $\rho(f(x_j), f(y_j)) \geq \epsilon$. Since $\mathcal{M}$ is sequentially compact, there is a subsequence $\{x_{j_\ell}\}_{\ell=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to a point $x \in \mathcal{M}$. Using the same sequence $\{j_\ell\}_{\ell=1}^\infty$ of indices, we get a subsequence $\{y_{j_\ell}\}_{\ell=1}^\infty$ of $\{y_j\}_{j=1}^\infty$.

We claim that $\{y_{j_\ell}\}_{\ell=1}^\infty$ converges to x in $\mathcal{M}$. To see this, we use the triangle inequality to get that $d(x, y_{j_\ell}) \leq d(x, x_{j_\ell}) + d(x_{j_\ell}, y_{j_\ell})$ for each $\ell \in \mathbb{Z}^+$. This implies that $d(x, y_{j_\ell}) < d(x, x_{j_\ell}) + \dfrac{1}{j_\ell} \leq d(x, x_{j_\ell}) + \dfrac{1}{\ell}$ for every $\ell \in \mathbb{Z}^+$. Then $\lim\limits_{\ell\to\infty} d(x, y_{j_\ell}) = 0$ because $\lim\limits_{\ell\to\infty} d(x, x_{j_\ell}) = 0$ and $\lim\limits_{\ell\to\infty} \dfrac{1}{\ell} = 0$.

It follows that $\{f(x_{j_\ell})\}_{\ell=1}^\infty$ and $\{f(y_{j_\ell})\}_{\ell=1}^\infty$ both converge to $f(x)$ in $\mathcal{N}$ because f is continuous at x. Note that $\epsilon \leq \rho(f(x_{j_\ell}), f(y_{j_\ell})) \leq \rho(f(x_{j_\ell}), f(x)) + \rho(f(x), f(y_{j_\ell}))$ for every $\ell \in \mathbb{Z}^+$. We can use this to get a contradiction since $\lim\limits_{\ell\to\infty} f(x_{j_\ell}) = \lim\limits_{\ell\to\infty} f(y_{j_\ell}) = f(x)$ in $\mathcal{N}$. □

Example 33.1 (Continuous But Not Uniformly Continuous)

The real-valued function f on $\mathbb{R}$ defined by $f(x) = x^2$ for each $x \in \mathbb{R}$.

Sketch of proof. Note that $f(x) - f(y) = (x + y)(x - y)$ for all $x, y \in \mathbb{R}$. We can use this to check that f is not uniformly continuous on $\mathbb{R}$, i.e. by fixing $x - y$ and letting $x + y$ grow without bound. □

Example 33.2 (Continuous But Not Uniformly Continuous)

$f : (0, 1] \to \mathbb{R}$ with $f(x) = \dfrac{1}{x}$ for each $x \in (0, 1]$ defined on the metric spaces $\mathcal{M} = (0, 1]$ and $\mathcal{N} = \mathbb{R}$, using the Euclidean metric on $\mathbb{R}$ restricted to $(0, 1]$.

Sketch of proof. It is easy to see that f is continuous on $(0, 1]$. Note that $f(x) - f(y) = \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{y - x}{xy}$ for every $x, y \in (0, 1]$. We can use this to check that f is not uniformly continuous, i.e. by fixing $y - x$ and letting $\dfrac{1}{xy}$ grow without bound. □

Theorem 33.3 (Continuity Preserves Connectedness)

Let f be a continuous mapping from $\mathcal{M}$ into $\mathcal{N}$. If E is a connected subset of $\mathcal{M}$, then $f(E)$ is connected in $\mathcal{N}$.

Proof. Suppose, for the sake of contradiction, that $f(E)$ is not connected in $\mathcal{N}$. This means that there are nonempty separated subsets A and B of $\mathcal{N}$ such that $A \cup B = f(E)$. Let $A_1 = f^{-1}(A) \cap E$ and $B_1 = f^{-1}(B) \cap E$. Thus, $A_1, B_1 \subseteq E$, by construction, so that $A_1 \cup B_1 \subseteq E$. We can check that $E \subseteq A_1 \cup B_1$ because $f(E) = A \cup B$. More precisely, if $x \in E$, then $f(x) \in f(E) = A \cup B$, so that $f(x) \in A$ or $f(x) \in B$. This implies that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, and hence $x \in A_1$ or $x \in B_1$ because $x \in E$. Thus, $A_1 \cup B_1 = E$. We will show that A_1 and B_1 are separated in $\mathcal{M}$.

We can check that $f(A_1) = A$ and $f(B_1) = B$ by construction. Note that $f(\overline{A_1}) \subseteq \overline{f(A_1)}$ and $f(\overline{B_1}) \subseteq \overline{f(B_1)}$. In particular, this follows from $A, B \subseteq f(E)$. Then $f(\overline{A_1}) \subseteq \overline{A}$ and $f(\overline{B_1}) \subseteq \overline{B}$. It follows that $f(\overline{A_1} \cap B_1) \subseteq f(\overline{A_1}) \cap f(B_1) \subseteq \overline{A} \cap B = \emptyset$, by the assumption that A and B are separated in $\mathcal{N}$. This implies that $\overline{A_1} \cap B_1 = \emptyset$. We can verify that $A_1 \cap \overline{B_1} = \emptyset$ similarly. This shows that A_1 and B_1 are separated in $\mathcal{M}$. Let us now check that $A_1, B_1 \neq \emptyset$. Note that A, B are nonempty by hypothesis. Because $A, B \subseteq f(E)$, it follows that there are $x, y \in E$ such that $f(x) \in A$ and $f(y) \in B$. This implies that $x \in A_1$ and $y \in B_1$, so that $A_1, B_1 \neq \emptyset$. This shows that E is not connected in $\mathcal{M}$, as desired. □

§34 November 18, 2019

Corollary 34.1 (Intermediate Value Theorem)

Let a, b be real numbers with $a < b$ and let f be a continuous real-valued function on $[a, b]$ using the standard Euclidean metric in $\mathbb{R}$ restricted to $[a, b]$. If $t \in \mathbb{R}$ satisfies $f(a) < t < f(b)$ or $f(b) < t < f(a)$, then there is a $z \in [a, b]$, then there is a $z \in [a, b]$ such that $f(z) = t$.

Proof. Let $f(x) = f(a)$ for every $x \in \mathbb{R}$ with $x < a$ and $f(y) = f(b)$ for every $y \in \mathbb{R}$ with $y > b$. We can check that this defines f as a continuous real-valued function on $\mathbb{R}$. We have that $[a, b]$ is connected as a subset of $\mathbb{R}$. This implies that $f([a, b])$ is connected in $\mathbb{R}$, by the above theorem. It follows that $t \in f([a, b])$, by connectedness, as desired. □

Definition 34.2. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of mappings from $\mathcal{M}$ into $\mathcal{N}$ and let f be another mapping from $\mathcal{M}$ into $\mathcal{N}$. We say that $\{f_j\}_{j=1}^{\infty}$ converges to f **pointwise** on $\mathcal{M}$ if for every $x \in \mathcal{M}$, we have that $\{f_j(x)\}_{j=1}^{\infty}$ converges to $f(x)$ in $\mathcal{N}$. We say that $\{f_j\}_{j=1}^{\infty}$ converges to f **uniformly** on $\mathcal{M}$ if for every $\epsilon > 0$, there is an $L \in \mathbb{Z}^+$ such that for every $j \geq L$ and $x \in \mathcal{M}$, we have that $\rho(f_j(x), f(x)) < \epsilon$.

Remark 34.3. Uniform convergence implies pointwise convergence.

Example 34.4

$\mathcal{M} = [0, 1]$ and $\mathcal{N} = \mathbb{R}$ with the standard Euclidean metric with $f_j(x) = x^j$ for each $j \in \mathbb{Z}^+$ and $x \in [0, 1]$ is pointwise convergent but not uniformly convergent.

Proof. Note that $\lim\limits_{j\to\infty} f_j(x) = 0$ when $0 \leq x < 1$ and 1 when $x = 1$, implying pointwise convergent. However, we can check that $\{f_j\}_{j=1}^{\infty}$ does not converge uniformly on $\mathcal{M}$. □

§35 November 20, 2019

Theorem 35.1 (Limit of Continuous Functions Is Continuous)

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of continuous mappings from $\mathcal{M}$ into $\mathcal{N}$ that converges uniformly to a mapping f from $\mathcal{M}$ into $\mathbb{N}$. Then f must be continuous on $\mathcal{M}$.

Proof. Let a point $x \in \mathcal{M}$ be given. We show that f is continuous at x. Let $\epsilon > 0$ be given. Since $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on $\mathcal{M}$, there is an $L \in \mathbb{Z}^+$ such that for every $j \geq L$ and $y \in \mathcal{M}$, we have that $\rho(f_j(y), f(y)) < \frac{\epsilon}{3}$. Since f_L is continuous at x, there is a $\delta_L > 0$ such that for every $y \in \mathcal{M}$ with $d(x, y) < \delta_L$, we have that $\rho(f_L(x), f_L(y)) < \frac{\epsilon}{3}$. Note that $\rho(f(x), f(y)) \leq \rho(f(x), f_L(y)) + \rho(f_L(x), f_L(y)) + \rho(f_L(y), f(y))$ for every $y \in \mathcal{M}$ using the triangle inequality twice. If $d(x, y) < \delta_L$, then we get that $\rho(f(x), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, as desired. □

Proposition 35.2 (Weierstrass' M-Test)

Let E be a nonempty set, and let $a_1(x), a_2(x), a_3(x), \ldots$ be an infinite sequence of complex-valued functions on E. Suppose that for each $j \in \mathbb{Z}^+$, there is a nonnegative real number A_j such that $|a_j(x)| \leq A_j$ for each $x \in E$, and that $\sum_{j=1}^{\infty} A_j$ converges in $\mathbb{R}$.[a] Then the sequence of partial sums $\left\{\sum_{j=1}^{n} a_j(x)\right\}_{n=1}^{\infty}$ converges to $\sum_{j=1}^{\infty} a_j(x)$ uniformly on E.

[a] Note that the condition of convergence implies that for each $x \in E$, $\sum_{j=1}^{\infty} a_j(x)$ converges absolutely as an infinite series of complex numbers, by the comparison test.

Proof. If $n \in \mathbb{Z}^+$, then $\sum_{j=1}^{\infty} a_j(x) - \sum_{j=1}^{n} a_j(x) = \sum_{j=n+1}^{\infty} a_j(x)$. Thus,

$$\left|\sum_{j=1}^{\infty} a_j(x) - \sum_{j=1}^{n} a_j(x)\right| = \left|\sum_{j=n+1}^{\infty} a_j(x)\right|.$$

We can check that $\left|\sum_{j=n+1}^{\infty} a_j(x)\right| \leq \sum_{j=n+1}^{\infty} |a_j(x)|$. More precisely, if $r \in \mathbb{Z}^+$ and $r \geq n+1$, then $\left|\sum_{j=n+1}^{r} a_j\right| \leq \sum_{j=n+1}^{r} |a_j(x)|$ by the triangle inequality. It follows that $\left|\sum_{j=n+1}^{r} a_j\right| \leq \sum_{j=n+1}^{\infty} |a_j(x)|$ because $\sum_{j=n+1}^{r} |a_j(x)| \leq \sum_{j=n+1}^{\infty} |a_j(x)|$.

Note that $\sum_{j=n+1}^{\infty} |a_j(x)| \leq \sum_{j=n+1}^{\infty} A_j$ for every $n \in \mathbb{Z}^+$ and $x \in E$ because $|a_j(x)| \leq A_j$ for every $x \in E$ and $j \in \mathbb{Z}^+$ by assumption. It follows that $\left|\sum_{j=1}^{\infty} a_j(x) - \sum_{j=1}^{n} a_j(x)\right| \leq \sum_{j=n+1}^{\infty} A_j$ for each $x \in E$ and $n \in \mathbb{Z}^+$. Note that $\sum_{j=n+1}^{\infty} A_j = \sum_{j=1}^{\infty} A_j - \sum_{j=1}^{n} A_j \to 0$ as $n \to \infty$, because $\sum_{j=1}^{\infty} A_j$ converges, by assumption.

Let $\epsilon > 0$ be given. By the previous statement, there is an $L \in \mathbb{Z}^+$ such that $\sum_{j=n+1}^{\infty} A_j < \epsilon$ for every $n \geq L$. Combining this with the above result, we get that $\left|\sum_{j=1}^{\infty} a_j(x) - \sum_{j=1}^{n} a_j(x)\right| < \epsilon$ for every $x \in E$ and $n \geq L$, as desired. □

§36 November 22, 2019

Proposition 36.1 (Power Series to Continuous Function)

Let $\sum_{j=0}^{\infty} a_j z^j$ be a power series with coefficients in $\mathbb{C}$. Let r be a nonnegative real number such that $\sum_{j=0}^{\infty} |a_j| r^j$ converges.[a] Then $f(z) = \sum_{j=0}^{\infty} a_j z^j$ defines a continuous complex-valued function on $S = \{z \in \mathbb{C} \mid |z| \leq r\}$, with respect to the standard Euclidean metric on $\mathbb{C}$ and its restriction to S.

[a] If $z \in \mathbb{C}$ and $|z| \leq r$, then it follows that $\sum_{j=0}^{\infty} a_j z^j$ converges absolutely by the comparison test.

Proof. Note that the sequence of partial sums $\{\sum_{j=0}^{n} a_j z^j\}_{n=0}^{\infty}$ converges to f uniformly on S, by Weirstrass (Proposition 35.2). For each $n \geq 0$, $\sum_{j=0}^{n} a_j z^j$ is continuous on $\mathbb{C}$. This implies that the restriction of $\sum_{j=0}^{n} a_j z^j$ to S is continuous. It follows that f is continuous on S too, by Theorem 35.1. □

Proposition 36.2

Suppose that R is a positive extended real number such that $\sum_{j=0}^{\infty} a_j z^j$ converges for every $z \in \mathbb{C}$ with $|z| < R$. Then $f(z) = \sum_{j=0}^{n} a_j z^j$ is a continuous complex-valued function on $S = \{z \in \mathbb{C} \mid |z| < R\}$ with respect to the standard Euclidean metric on $\mathbb{C}$ and its restriction to S.

Proof. If $0 \leq r < R$, then $\sum_{j=0}^{\infty} |a_j| r^j$ converges, by the previous proposition. More precisely, we can choose $w \in \mathbb{C}$ such that $r < |w| < R$ to get that $\sum_{j=0}^{\infty} a_j w^j$ converges by assumption. This implies that $\sum_{j=0}^{\infty} |a_j| r^j$ converges by an earlier proposition.

Let $z_0 \in S$ be given. We will show that f is continuous at z_0, as a complex-valued function on S. Because $|z_0| < R$, there is an $r \in \mathbb{R}$ such that $|z_0| < r < R$. Thus,

$\sum_{j=0}^{\infty} |a_j| r^j$ converges, as before. By the previous proposition, we get that the restriction of f to $S' = \{z \in \mathbb{C} \mid |z| \leq r\}$ is continuous. In particular, the restriction of f to S' is continuous at z_0, which is an element of S'. We can use this and the fact that $|z_0| < r$, by construction, to get that f is continuous at z_0 as a function on S. □

§36.1 Bounded Mapping

Definition 36.3. Let $\mathcal{M}$ be a set and let $(\mathcal{N}, \rho(u,v))$ be a metric space. A mapping f from $\mathcal{M}$ into $\mathcal{N}$ is **bounded** if $f(\mathcal{M})$ is bounded as a subset of $\mathcal{N}$. Suppose now that $(\mathcal{M}, d(x,y))$ is also a metric space. Let $C_b(\mathcal{M}, \mathcal{N})$ be the space of all mappings f from $\mathcal{M}$ into $\mathcal{N}$ that are both bounded and continuous.

Remark 36.4. If $\mathcal{M}$ is compact, then every continuous mapping f from $\mathcal{M}$ into $\mathcal{N}$ is bounded. More precisely, $f(\mathcal{M})$ is a compact subset of $\mathcal{N}$, which implies that $f(\mathcal{M})$ is a bounded subset of $\mathcal{N}$.

Remark 36.5. Suppose that $f, g \in C_b(\mathcal{M}, \mathcal{N})$. Because $f(\mathcal{M})$ and $g(\mathcal{M})$ are bounded subsets of $\mathcal{N}$, there are $p, q \in \mathcal{N}$ and $r, t > 0$ such that $\rho(f(x), p) < r$ and $\rho(g(x), q) < t$ for every $x \in \mathcal{M}$. This implies that $\rho(f(x), g(x)) \leq \rho(f(x), p) + \rho(p, q) + \rho(q, g(x)) < r + \rho(p,q) + t$ for every $x \in \mathcal{M}$. It follows that $\rho(f(x), g(x))$ is bounded as a real-valued function on $\mathcal{M}$ with respect to the standard Euclidean metric on $\mathbb{R}$.

Definition 36.6. $\Theta(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in \mathcal{M}\}$, which is a nonnegative real number.

Remark 36.7. If $f = g$ on $\mathcal{M}$, then $\rho(f(x), g(x)) = 0$ for every $x \in \mathcal{M}$, and $\Theta(f, g) = 0$. Conversely, if $\Theta(f, g) = 0$, then $\rho(f(x), g(x)) = 0$ for every $x \in \mathcal{M}$. This implies that $f(x) = g(x)$ for every $x \in \mathcal{M}$, so that $f = g$ on $\mathcal{M}$.

Remark 36.8. $\Theta(f, g) = \Theta(g, f)$ because $\rho(u, v) = \rho(v, u)$ for every $u, v \in \mathcal{N}$.

Definition 36.9. Let $f, g, h \in C_b(\mathcal{M}, \mathcal{N})$ be given. Note that $\rho(f(x), h(x)) \leq \rho(f(x), g(x)) + \rho(g(x), h(x))$ for every $x \in \mathcal{M}$ by the triangle inequality. It follows that $\rho(f(x), h(x)) \leq \Theta(f, g) + \Theta(g, h)$ for every $x \in \mathcal{M}$ by definition of Θ. This implies that $\Theta(f, h) \leq \Theta(f, g) + \Theta(g, h)$, by the definition of supremum. Thus, Θ defines a metric on $c_b(\mathcal{M}, \mathcal{N})$, which is called the **supremum metric**.

§37 November 25, 2019

Proposition 37.1

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of elements of $C_b(\mathcal{M}, \mathcal{N})$, and let f be another element of $C_b(\mathcal{M}, \mathcal{N})$. Then $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to $\Theta(\cdot, \cdot)$ if and only if $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on $\mathcal{M}$.

Proof. Suppose that $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to Θ, and let $\epsilon > 0$ be given. Thus, there is an $L \in \mathbb{Z}^+$ such that $\Theta(f_j, f) < \epsilon$ for every $j \geq L$. This implies that $\rho(f_j(x), f(x)) \leq \Theta_j(f_j, f) < \epsilon$ for every $x \in \mathcal{M}$ and $j \geq L$, as desired.

Suppose now that $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on $\mathcal{M}$. Let $\eta > 0$ be given. By the definition of uniform convergence, there is an $L(\eta) \in \mathbb{Z}^+$ such that $\rho(f_j(x), f(x)) < \eta$ for every $x \in \mathcal{M}$ and $j \geq L(\eta)$. This implies that $\Theta(f_j, f) \leq \eta$ for every $j \geq L(\eta)$. If $\epsilon > 0$

is arbitrary, then we can apply this to any $\eta \in (0, \epsilon)$ (such as $\frac{\epsilon}{2}$) to get that $\Theta(f_j, f) < \epsilon$ for every $j \geq L(\epsilon)$, as desired. □

Theorem 37.2

If $\mathcal{N}$ is complete as a metric space with respect to $\rho(\cdot, \cdot)$, then $C_b(\mathcal{M}, \mathcal{N})$ is complete with respect to $\Theta(\cdot, \cdot)$.

Proof. Let $\{f_j\}_{j=1}^{\infty}$ be an arbitrary Cauchy sequence in $C_b(\mathcal{M}, \mathcal{N})$. This means that for each $\epsilon > 0$, there is an $L(\epsilon) \in \mathbb{Z}^+$ such that $\Theta(f_j, f_\ell) < \epsilon$ for every $j, \ell \geq L(\epsilon)$. This implies that $\rho(f_j(x), f_\ell(x)) < \epsilon$ for every $x \in \mathcal{M}$ and $j, \ell \geq L(\epsilon)$. If $x \in \mathcal{M}$, then it follows that $\{f_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$. Since $\mathcal{N}$ is complete by assumption, then we get that $\{f_j\}_{j=1}^{\infty}$ converges to an element of $\mathcal{N}$. For each $x \in \mathcal{M}$, let $f(x)$ be the limit of this sequence in $\mathcal{N}$, i.e., $f(x) = \lim_{j\to\infty} f_j(x)$. This defines f as a function on $\mathcal{M}$ with values in $\mathcal{N}$. Note that $\{f_j\}_{j=1}^{\infty}$ converges to f pointwise on $\mathcal{M}$, by construction. We can then check that $\rho(f_j(x), f(x)) \leq \epsilon$ for every $x \in \mathcal{M}$ and $j \geq L(\epsilon)$. This implies that $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on $\mathcal{M}$. It follows that f is continuous on $\mathcal{M}$, by a previous theorem, because f_j is continuous on $\mathcal{M}$ for each $j \in \mathbb{Z}^+$ by assumption. We can also check that f is a bounded function on $\mathcal{M}$. More precisely, this can be obtained by taking $\epsilon = 1$, using the fact that $f_L(1)$ is bounded on $\mathcal{M}$, by assumption. Thus, $f \in C_b(\mathcal{M}, \mathcal{N})$. Using the previous proposition, we get that $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to $\Theta(\cdot, \cdot)$ □

§38 Thanksgiving

§39 December 2, 2019

Definition 39.1. Let E be a subset of $\mathcal{M}$ and $p \in \mathcal{M}$ be a limit point of E in $\mathcal{M}$. Let f be a function defined on E with values in $\mathcal{N}$, and let q be an element of $\mathcal{N}$. We say that **$f(x)$ tends to q** in $\mathcal{N}$ as $x \in E$ tends to $p \in \mathcal{M}$, or $\lim_{x\to p} f(x) = q$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $x \in E$ with $d(x, p) < \delta$ and $x \neq p$, then $\rho(f(x), q) < \epsilon$.

Remark 39.2. p may or may not be an element of E. If $p \in E$, then the value of f at p is not used in the definition of the limit.

Proposition 39.3

If $q \in \mathcal{M}$, then $\lim_{x\in E\to p} f(x) = q$ if and only if for every sequence $\{x_j\}_{j=1}^{\infty}$ of elements of E such that for every $j \in \mathbb{Z}^+$, $x_j \neq p$, and $\{x_j\}_{j=1}^{\infty}$ converges to p in $\mathcal{M}$, we have that $\{f(x_j)\}_{j=1}^{\infty}$ converges to q in $\mathcal{N}$.

Sketch of proof. This can be shown basically the same way as for the analogous characterization of continuity of a mapping from $\mathcal{M}$ into $\mathcal{N}$ at p. □

Remark 39.4. If $p \in \mathcal{M}$ is a limit point of a set $E \subseteq \mathcal{M}$, then one can check that there is a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of E that converges to p in $\mathcal{M}$ and satisfies $x_j \neq p$ for each $j \in \mathbb{Z}^+$.

Remark 39.5. Using the above remark and proposition, we get that $\lim_{x \to p} f(x)$ is unique when it exists, because of the analogous statement for convergent sequences in $\mathcal{N}$.

Proposition 39.6

Let E be a subset of $\mathcal{M}$ and suppose that $p \in \mathcal{M}$ is a limit point of E. Let f and g be complex-valued functions on E and suppose that $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$ for some $A, B \in \mathbb{C}$, using the standard Euclidean metric on $\mathbb{C}$. $\lim_{x \to p}(f(x)+g(x)) = A+B$ and $\lim_{x \to p}(f(x)g(x)) = AB$. If we also have that for each $x \in E$, $g(x) \neq 0$, and $B \neq 0$, then $\lim_{x \to p} \frac{1}{g(x)} = \frac{1}{B}$.

Proof. This follows from the previous proposition and the analogous statements for convergent sequences of complex numbers. □

Remark 39.7. Let f be a mapping from $\mathcal{M}$ into $\mathcal{N}$. Let $p \in \mathcal{M}$ be given. If p is a limit point of $\mathcal{M}$, then one can check that f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$. Otherwise, if p is not a limit point of $\mathcal{M}$, then there is an $r > 0$ such that the only $x \in \mathcal{M}$ with $d(p, x) < r$ is $x = p$. In this case, one can check that any mapping from $\mathcal{M}$ into $\mathcal{N}$ is continuous at p.

Definition 39.8. Let a and b be real numbers with $a < b$. Let f be a function defined on $E = (a, b)$ with values in $\mathcal{N}$. if $p \in [a, b) \subseteq (a, b)$, then let $E_+(p) = (p, b)$. Note that p is a limit point of $E_+(p)$ with respect to the standard Euclidean metric on $\mathbb{R}$. If $\lim_{x \to p} f(x)$ exists in $\mathcal{N}$, then it may be denoted $\lim_{x \to p^+} f(x)$ or $f(p+)$. Similarly, if $p \in (a, b]$, then let $E_-(p) = (a, p)$. If $\lim_{x \to p} f(x)$ exists in $\mathcal{N}$, then it may be denoted $\lim_{x \to p^-} f(x)$ or $f(p-)$.

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Remark 40.1. We can check that $\lim_{x \to p} f(x)$ exists using the standard metric on $\mathbb{R}$ and its restriction to (a, b) if and only if the one-sided limits $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ exist and are equal, in which case $\lim_{x \to p} f(x) = \lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x)$. In particular, f is continuous at p if and only if $\lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x) = f(p)$.

Definition 40.2. If $\lim_{x \to p^+} f(x) = f(p)$, then f is **right continuous** at p. Similarly, if $\lim_{x \to p^-} f(x) = f(p)$, then f is **left continuous**. Thus, f is continuous at p if and only if f is both left and right continuous at p.

Theorem 40.3

Let f be a real-valued monotonically increasing function on (a, b). Then for each $p \in (a, b)$, the one-sided limits $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ exist and satisfy $f(p-) \leq f(p) \leq f(p+)$. If $p, q \in (a, b)$ and $p < q$, then $f(p+) \leq f(q-)$.

Proof. Let $p \in (a, b)$ be given. Let $A_+(p) = \{f(x) \mid p < x < b\}$ and $A_-(p) = \{f(x) \mid a < x < p\}$. Observe that $A_+(p), A_-(p) \neq \emptyset$, $f(p)$ is a lower bound for $A_+(p)$, and $f(p)$ is an upper bound for $A_-(p)$. It follows that the infimum of $A_+(p)$ and supremum of $A_-(p)$ exist in $\mathbb{R}$, with $\inf A_+(p) \geq f(p)$ and $\sup A_-(p) \leq f(p)$.

We claim that $\lim_{x \to p^+} f(x) = \inf A_+(p)$ and $\lim_{x \to p^-} f(x) = \sup A_-(p)$. To check the first statement, let $\epsilon > 0$ be given. By definition of the infimum, $\inf A_+(p) + \epsilon$ is not a lower bound for $A_+(p)$, and hence there is an $x(\epsilon) \in (p, b)$ such that $f(x(\epsilon)) < \inf A_+(p) + \epsilon$. If $x \in (p, x(\epsilon)) \leq (p, b)$, then $\inf A_+(p) \leq f(x) \leq f(x(\epsilon)) < \inf A_+(p) + \epsilon$. It follows that $|f(x) - \inf A_+(p)| = f(x) - \inf A_+(p) < \epsilon$, as desired. (Take $\delta = x(\epsilon) - p$.) The proof of the second statement in the claim is similar, Note that $f(p-) \leq f(p) \leq f(p+)$ by construction. For the third statement, let $p, q \in (a, b)$ be given with $p < q$. Let x be any element of (p, q). One can check that $f(p+) \leq f(x) \leq f(q-)$ by the earlier characterizations of $f(p+), f(q-)$. □

Theorem 40.4

Let f be a monotonically increasing real-valued function on (a, b), and let A be the set of $p \in (a, b)$ such that f is not continuous at p. Then A has only finitely or countably many elements.

Proof. Observe that $A = \{p \in (a, b) \mid f(p-) < f(p+)\}$ by the previous theorem. If $p \in A$, then choose a rational number $r(p)$ in $(f(p-), f(p+))$. Suppose that $p, q \in A$ and $p < q$. Note that $f(p+) < f(q-)$. It follows that $r(p) < f(p+) \leq f(q-) < r(q)$, so that $r(p) < r(q)$. Note that $\{r(p) \mid p \in A\}$ has only finitely or countably many elements, because it is a subset of the set $\mathbb{Q}$ of all rational numbers, which we have shown to be countably infinite. One can check that A has only finitely or countably many elements. □

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§41.1 Path Connected

Definition 41.1. A subset E of $\mathcal{M}$ is **path connected** in $\mathcal{M}$ if for every pair of points p and q in E, there is a continuous mapping f from $[0, 1]$ into $\mathcal{M}$ such that $f(0) = p$ and $f(1) = q$, and $f([0, 1]) \subseteq E$. Here we use the restriction of the standard Euclidean metric on $\mathbb{R}$ to $[0, 1]$, as usual.

Proposition 41.2

If $E \subseteq \mathcal{M}$ is path connected, then E is connected.

Proof. Suppose for the sake of contradiction that E is not connected. This means that there is a nonempty separated subset A, B of $\mathcal{M}$ such that $E = A \cup B$. Let p be an element of A and q an element of B. Since E is path connected, there is a continuous mapping f from $[0, 1]$ into $\mathcal{M}$ such that $f(0) = p$, $f(1) = q$, and $f([0, 1]) \subseteq E$. Let $f(x) = f(0)$ when $x < 0$ and $f(x) = f(1)$ when $x > 1$. This defines f as a mapping from $\mathbb{R}$ into $\mathcal{M}$, and one can check that f is continuous on $\mathbb{R}$. It follows from previous results that $f([0, 1])$ is a connected subset of $\mathcal{M}$ because $[0, 1]$ is a connected subset of $\mathbb{R}$. Let $A_0 = A \cap f([0, 1])$ and $B_0 = B \cap f([0, 1])$. Note that $p \in A_0$ and $q \in B_0$, so that

$A_0, B_0 \neq \emptyset$. One can check that A_0 and B_0 are separated as subsets of $\mathcal{M}$, because A and B are separated and $A_0 \subseteq A$, $B_0 \subseteq B$, by construction. More precisely, one can verify that $\overline{A_0} \subseteq \bar{A}$ and $\overline{B_0} \subseteq \bar{B}$. We also have that $A_0 \cup B_0 = (A \cap f([0,1])) \cup (B \cap f([0,1])) = (A \cup B) \cap f([0,1]) = E \cap f([0,1]) = f([0,1])$, because $f([0,1]) \subseteq E$, by assumption. This contradicts the connectedness of $f([0,1])$ in $\mathcal{M}$, as desired. $\square$