MATH 102 Lecture Notes

Taught by Daniel Hast

Shaquille Que

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This is Rice University's MATH 102, instructed by Dr. Daniel Hast. The formal name for this class is "Single Variable Calculus II". These notes are essentially what were covered in class, but I have added my own comments when appropriate. This file was created in LaTeXand uses Evan Chen's `evan.sty` package. Any mistake herein is my own. Please let me know of any errors by emailing me at stq1@rice.edu.

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§1 August 22, 2018

§1.1 Integration by Substitution

Example 1.1

Integrate:

$$\int t^2 e^{5-t^3} dt.$$

Solution.

$$\begin{aligned}
&\int t^2 e^{5-t^3} dt \\
&= \int -\frac{1}{3} e^u du \\
&= -\frac{1}{3} e^u + C \\
&= -\frac{1}{3} e^{5-t^3} + C.
\end{aligned}$$

□

Example 1.2

Integrate:

$$\int t(2-t)^{57} dt.$$

Solution.

$$\begin{aligned}
&\int t(2-t)^{57} \\
&= \int -(2-u)u^{57} du \\
&= \int u^{58} - 2u^{57} du \\
&= \frac{u^{59}}{59} - 2\frac{u^{58}}{58} + C \\
&= \frac{(2-t)^{59}}{59} - 2\frac{(2-t)^{58}}{58} + C.
\end{aligned}$$

□

§1.2 Indefinite Integrals

The following are some common examples of indefinite integrals, more of which can be found in the book.

$$\begin{aligned}
\int \frac{dt}{\sqrt{a^2 - t^2}} &= \sin^{-1} \frac{t}{a} + C \\
\int \frac{dt}{a^2 + t^2} &= \frac{1}{a} \tan^{-1} \frac{t}{a} + C \\
\int \frac{dt}{t\sqrt{t^2 - a^2}} &= \frac{1}{a} \sec^{-1} \frac{t}{a} + C.
\end{aligned}$$

Example 1.3

Integrate:

$$\int \frac{dt}{\sqrt{5-(t-2)^2}}.$$

Solution.

$$\begin{aligned}&\int \frac{dt}{\sqrt{5-(t-2)^2}}\\ =&\int \frac{du}{\sqrt{\sqrt{5}^2-u^2}}\\ =&\sin^{-1}\frac{t-2}{\sqrt{5}}+C.\end{aligned}$$

□

§2 August 24, 2018

§2.1 Integration by Parts

Example 2.1

Evaluate:

$$\int x^2e^x\,dx.$$

Solution.

$$\begin{aligned}&\int x^2e^x\,dx\\ =&x^2e^x-\int e^x\,d(x^2)\\ =&x^2e^x-\int 2xe^x\,dx\\ =&x^2e^x-(2xe^x-\int 2e^x\,dx)\\ =&x^2e^x-2xe^x+2e^x+C.\end{aligned}$$

□

Example 2.2

Evaluate:

$$\int 3t^5\sin(t^3)\,dt.$$

Solution.

$$\begin{aligned}
&\int 3t^5 \sin(t^3)\, dt \\
&= \int u \sin u \, du \\
&= \int -u\cos u - \int -\cos u \\
&= -\, u\cos u + \sin u + C \\
&= -\, t^3 \cos t^3 + \sin t^3 + C.
\end{aligned}$$

□

where $u = t^3$.

Example 2.3

Evaluate:

$$\int x^n \ln(x)\, dx, \quad n \neq 1.$$

Solution.

$$\begin{aligned}
&\int x^n \ln(x)\, dx \\
&= \ln(x)\frac{x^{n+1}}{n+1} - \int \frac{1}{n+1} x^n \\
&= \ln(x)\frac{1}{n+1} x^{n+1} - \frac{1}{(n+1)^2} x^{n+1} + C.
\end{aligned}$$

□

§3 August 27, 2018

§3.1 Trigonometric Integrals

Example 3.1 (Base Case)

$$\int \sin^n(x)\cos(x)\, dx.$$

Solution. Use substitution $u = \sin(x) \implies du = \cos(x)dx$.

$$\begin{aligned}
&\int \sin^n(x)\cos(x)\, dx \\
&= \int u^n \, du = \frac{1}{n+1} u^{n+1} + C \\
&= \frac{1}{n+1} \sin^{n+1}(x) + C.
\end{aligned}$$

□

Exercise 3.2. $\cos^n(x)\sin(x)\, dx$

Example 3.3 (When an Exponent is Odd)

Evaluate:

$$\int \sin^2(x)\cos^3(x)\, dx.$$

Solution.

$$\begin{aligned}
&\int \sin^2(x)\cos^3(x)\,dx \\
=&\int \sin^2 x(\cos^2 x)\cos x\,dx \\
=&\int \sin^2 x(1-\sin^2 x)\cos x\,dx \\
=&\int (\sin^2 x-\sin^4 x)\cos x\,dx \\
=&\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C.
\end{aligned}$$

□

A strategy that won't work or will make the problem more complicated is using the substitution $\sin^2 x = 1-\cos^2 x$. Try this yourself.

Example 3.4 (Another Example)
Evaluate:

$$\int \sin^5 x\cos^4 x\,dx.$$

Solution.

$$\begin{aligned}
&\int \sin^5 x\cos^4 x\,dx \\
=&\int \sin x\sin^4 x\cos^4 x\,dx \\
=&\int \sin x(1-\cos^2 x)^2\cos^4 x\,dx \\
=&\int \sin x(1-2\cos^2 x+\cos^4 x)\cos^4 x\,dx \\
=&\int \sin x(\cos^4 x-2\cos^6 x+\cos^8 x)\,dx \\
=&-\frac{1}{5}\cos^5 x+\frac{2}{7}\cos^7 x-\frac{1}{9}\cos^9 x+C.
\end{aligned}$$

□

We also have the trigonometric identity $\tan^2 x+1=\sec^2 x$ (which can be easily derived from dividing both sides of the well-known identity $\sin^2 x+\cos^2 x=1$ by $\cos^2 x$).
Note that $\frac{d}{dx}\tan x=\sec^2 x$ and $\frac{d}{dx}\sec x=\sec x\tan x$.

Example 3.5 (Tangents and Secants)
Evaluate:

$$\int \sec^2 x\tan^3 x\,dx.$$

Solution.

$$\begin{aligned}
&\int \sec^2 x\tan^3 x\,dx \\
=&\int u^3\,du=\frac{1}{4}u^4+C=\frac{1}{4}\tan^4 x+C.
\end{aligned}$$

□

Example 3.6 (Another Example)

Evaluate:

$$\sec^4 x \tan^3 x \, dx.$$

Solution.

$$\begin{aligned} &\sec^4 x \tan^3 x \, dx \\ &= \int (1 + \tan^2 x) \sec^2 x \tan^3 x \, dx \\ &= \int \sec^2 x \tan^3 x \, dx + \tan^5 x \sec^2 x \, dx. \end{aligned}$$

□

Example 3.7 (Another Example)

Evaluate:

$$\int \sec^5 x \tan^3 x \, dx.$$

Solution.

$$\begin{aligned} &\int \sec^5 x \tan^3 x \, dx \\ &= \int \sec^5 x (\sec^2 x - 1) \tan x \, dx \\ &= \int \sec^4 x (\sec^2 x - 1) \sec x \tan x \, dx \\ &= \int u^4 (u^2 - 1) \, du. \end{aligned}$$

□

To handle when the exponents are all even, we also have the trigonometric identities $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$, $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$.

Exercise 3.8. $\displaystyle\int \sin^2 x \cos^4 x \, dx.$

§4 August 29, 2018

§4.1 Trigonometric Substitutions

Example 4.1

Evaluate:

$$\int \sqrt{9 - x^2} \, dx.$$

Solution.

$$
\begin{aligned}
&\int \sqrt{9-x^2}\,dx \\
=&\int 3\sqrt{1-\sin^2\theta}\cdot 3\cos\theta\,d\theta \\
=&\int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta \\
=&\int 9\cos^2\theta\,d\theta \\
=&9\int \frac{1+\cos 2\theta}{2}\,d\theta \\
=&9\int \frac{1}{2}\,d\theta + 9\int \frac{1}{2}\cos(2\theta)\,d\theta \\
=&\frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C \\
=&\frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{9}{4}\sin(2\sin^{-1}\frac{x}{3}) + C.
\end{aligned}
$$

Substitution: $x = 3\sin\theta \implies dx = 3\cos\theta\,d\theta$ and $\theta = \sin^{-1}\frac{x}{3}$.

Note that $-3 \le x \le 3 \implies -1 \le \frac{x}{3} \le 1$. □

Remark 4.2. Using the substitution $x = a\sin\theta$ is useful for terms of the form $\sqrt{a^2-x^2}$ because of the identity $\sin^2\theta + \cos^2\theta = 1$.

Example 4.3

Evaluate:

$$\int x^3\sqrt{1-x^2}\,dx.$$

Solution.

$$
\begin{aligned}
&\int x^3\sqrt{1-x^2}\,dx \\
=&\int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\
=&\int \sin^3\theta\cos^2\theta\,d\theta \\
=&\int \sin\theta\,(1-\cos^2\theta)\cos^2\theta\,d\theta \\
=&\int (-u^2+u^4)\,du \\
=&-\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\
=&-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C \\
=&-\frac{1}{3}\cos^3(\sin^{-1}x) + \frac{1}{5}\cos^5(\sin^{-1}x) + C \\
=&-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{1}{5}(1-x^2)^{\frac{5}{2}} + C.
\end{aligned}
$$

Substitution: $x = \sin\theta \implies dx = \cos\theta\, d\theta$ and $\theta = \sin^{-1} x$.
$u = \cos\theta \implies du = -\sin\theta\, d\theta$. □

Remark 4.4. We can also use the substitution $x = a\tan\theta$ for expressions of the form $\sqrt{a^2 + x^2}$ because of the identity $1 + \tan^2\theta = \sec^2\theta$.

Example 4.5

Evaluate:

$$\int \frac{1}{\sqrt{1+x^2}}\, dx.$$

Solution.

$$\begin{aligned}
&\int \frac{1}{\sqrt{1+x^2}}\, dx \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln|\sqrt{1+\tan^2\theta} + \tan\theta| \\
&= \ln|\sqrt{1+x^2} + x| + C.
\end{aligned}$$

□

These substitutions can be summarized under the following table:

Term	Identity	Substitution
$a^2 - x^2$	$1 - \sin^2\theta = \cos^2\theta$	$x = a\sin\theta$
$a^2 + x^2$	$1 + \tan^2\theta = \sec^2\theta$	$x = a\tan\theta$
$x^2 - a^2$	$\sec^2\theta - 1 = \tan^\theta$	$x = a\sec\theta$

§5 August 31, 2018

Example 5.1 (Trigonometric Substitution)

Evaluate:

$$\int \sqrt{x^2 + 2x - 3}\, dx.$$

Solution.

$$\begin{aligned}
&\int \sqrt{x^2 + 2x - 3}\, dx \\
&= \int \sqrt{(x+1)^2 - 4}\, dx \\
&= \int \sqrt{4\sec^2\theta - 4}\sec\theta\tan\theta\, d\theta \\
&= \int 4\tan\theta\sec\theta\tan\theta\, d\theta
\end{aligned}$$

where $x + 1 = 2\sec\theta \implies dx = 2\tan\theta\sec\theta\, d\theta$. □

§5.1 Partial Fractions

Example 5.2 (A Simple Example)
Evaluate
$$\int \frac{x^3+2x^2+3x+5}{x+1}\,dx.$$

Solution.
$$\begin{aligned}&\int \frac{x^3+2x^2+3x+5}{x+1}\,dx\\ =&\int (x^2+x+2+\frac{3}{x+1})\,dx\\ =&\frac{1}{3}x^3+\frac{1}{2}x^2+2x+3\ln\mid x+1\mid+C.\end{aligned}$$
□

In the above example, the denominator was of degree 1, so it was relatively simple to integrate. However, for denominators with degree greater than 1, we need to use the following theorem to integrate more easily.

Theorem 5.3 (Partial Fractions)
Suppose $\deg P < \deg Q$.
If $Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$ with all linear factors distinct, then one can find constants $A_1, A_2, \ldots, A_n$ such that
$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\ldots+\frac{A_n}{a_nx+b_n}.$$

Example 5.4 (Partial Fractions in Action)
Decompose the following fraction into partial fractions:
$$\frac{3x+2}{x^3-x^2-2x}$$

Solution.
$$\begin{aligned}\frac{3x+2}{x^3-x^2-2x}&=\frac{3x+2}{x(x+1)(x-2)}\\ &=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}\end{aligned}$$

Now, we find the common denominator and compare the numerators.
$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}=\frac{A(x+1)(x-2)+Bx(x-2)+Cx(x+1)}{x(x+1)(x-2)}$$
$$3x+2=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$$

Method 1: Substitution
$$\begin{aligned}x=0 &\implies 2=A(1)(-2)\\ x=-1 &\implies -1=B(-1)(-3)\\ x=2 &\implies 8=C(2)(3)\end{aligned}$$

Method 2: Comparing Coefficients
$$\begin{aligned}3x+2=(A+B+C)x^2+(-A-2B+C)x-2A\\ \implies A+B+C=0\\ -A-2B+C=3\\ -2A=2\end{aligned}$$

Regardless of which method we use, we'll obtain $A = -1$, $B = -\frac{1}{3}$, $C = \frac{4}{3}$, so

$$\frac{3x+2}{x^3-x^2-2x} = -\frac{1}{x} - \frac{1}{3(x+1)} + \frac{4}{3(x-2)}. \quad \square$$

§6 September 5, 2018

Complications with Partial Fractions

1. **Repeated linear factor**:

$$\frac{P(x)}{(cx+d)^n R(x)} = \frac{A_1}{cx+d} + \frac{A_2}{(cx+d)^2} + \cdots + \frac{A_n}{(cx+d)^n} + \text{ other terms from } R.$$

2. **Irreducible quadratic factor**:

$$\frac{P(x)}{(ax^2+bx+c)R(x)} = \frac{Ax+B}{ax^2+bx+c} + \text{ other terms from } R.$$

3. **Repeated irreducible quadratic factor**:

$$\frac{P(x)}{(ax^2+bx+c)^n R(x)} = \frac{A_1}{ax^2+bx+c} + \frac{A_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_n}{(ax^2+bx+c)^n}$$
$$+ \text{ other terms from } R.$$

This covers all cases, because any polynomial with degree at least 3 can actually be decomposed into linear and quadratic factors with real coefficients.

Example 6.1 (Decomposing into Partial Fractions)
Decompose the following into partial fractions:

$$\frac{25}{x^4-x^2-2x+2}$$

Solution. We use the Rational Root Theorem and long division to obtain that

$$x^4 - x^2 - 2x + 2 = (x-1)(x^3+x^2-2) = (x-1)^2(x^2+2x+2)$$

at which point we have that x^2+2x+2 is irreducible, because the discriminant $2^2 - 4(1)(2) = -4$ is negative. (We can also check that $x^2+2x+2 = (x+1)^2+1$, so that the parabola never touches the $x-$axis.) This means that the roots of the quadratic are imaginary, so it is irreducible.

$$\begin{aligned}\frac{25}{x^4-x^2-2x+2} &= \frac{25}{(x-1)^2(x^2+2x+2)} \\ &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2}\end{aligned}$$

So $25 = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x-1)^2$
At this point, we try out random[1] values for x to obtain several equations.

[1]Not exactly random, but does take some guesswork

$x = 1 \implies 25 = 5B \implies B = 5$
At this point, no other obvious substitutions come to mind, because no other factor can conveniently become 0, so we try a new strategy – we compare coefficients.
$0x^3 = Ax^3 + Cx^3 \implies 0 = A + C$
Repeating this, we can obtain
$A + B - 2C + D = 0$
$2B + C - 2D = 0$
$-2A + 2B - 2D$
and now, we have four equations in four variables. At this point, we can do the algebra[2] and obtain $(A, B, C, D) = (-4, 5, 4, 7)$. So

$$\frac{25}{x^4 - x^2 - 2x + 2} = -\frac{4}{x-1} + \frac{5}{(x-1)^2} + \frac{4x+7}{x^2+2x+2}. \quad \square$$

Now we move on to integrating partial fractions.

§6.1 Integrating Partial Fractions

The reason why it is important to break down fractions into partial fractions is that the latter is a lot easier to integrate, i.e. if the denominator is linear or quadratic, then it is easy to use either u-substitution or trig substitution to integrate.

1. **Linear Denominator:** $\displaystyle\int -\frac{4}{x-1}\,dx = -4\ln|x-1| + C.$

2. **Perfect Square Denominator:** $\displaystyle\int \frac{5}{(x-1)^2}\,dx = -\frac{5}{x-1} + C$
   You can come up with this by using the substitution $u = x - 1$.

3. **Non-Perfect Square Quadratic Denominator:**

$$\int \frac{4x+7}{x^2+2x+2}\,dx = \int \frac{4x+7}{(x+1)^2+1}\,dx = \int \frac{4u+3}{u^2+1}\,du$$

   Note that in this form, we can apply the trig substitution $u = \tan\theta$ to finish the integration.

§7 September 7, 2018

§7.1 Numerical Integration

Midpoint Rule

$$\int_a^b f(x)\,dx$$

If we use n partitions and
$M = \max\limits_{a \le x \le b} |f''(x)|$
then the error in midpoint rule $\le \dfrac{M(b-a)^2}{24n^2}$

Another method that can be used is the trapezoidal rule, which estimates the area under a curve using a trapezoid instead of rectangles in the midpoint rule.

[2] Do this yourself!

§7.2 Improper Integrals

Let $f : \mathbb{R} \to \mathbb{R}$. Define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$

We say that the integral **converges** if the limit exists and **diverges** otherwise.
If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ both converge, define

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 7.1

Find the area under the curve $y = \frac{1}{x}$ from $x = 1$ to ∞. Find also the volume of the volume of the solid generated by rotating this area about the $x-$axis.

Solution.

$$\begin{aligned}
&\int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} (\ln|x|)_1^t \\
&= \lim_{t\to\infty} (\ln(t) - \ln(1)) \\
&= \lim_{t\to\infty} \ln(t) \\
&= +\infty
\end{aligned}$$

so $\int_1^\infty \frac{1}{x}\,dx$ diverges. This means that if we take the area under the curve $y = \frac{1}{x}$, we will not be able to evaluate it.

$$\begin{aligned}
&\text{Volume of } \frac{1}{x} \text{ rotated around } x\text{-axis} \\
&= \int_1^\infty (\text{area of slice at } x)\,dx \\
&= \int_1^\infty \frac{\pi}{x^2}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,dx \\
&= \lim_{t\to\infty} \left(-\frac{\pi}{x}\right)_1^t \\
&= \lim_{t\to\infty} \left(-\frac{\pi}{t} + \pi\right) \\
&= \pi
\end{aligned}$$

□

This is surprising. Even though the area was infinite, the volume is actually finite! This is called Gabriel's Horn.

§8 September 10, 2018

Definition 8.1. If f is continuous on $[a, b)$, $\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{t \nearrow b} \int_a^t f(x)\,dx$.

Definition 8.2. If f is continuous on $(a, b]$, $\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{t \searrow a} \int_t^b f(x)\,dx$.

Definition 8.3. If f is continuous on $[a, b]$ except at c,
$\int_a^b f(x)\,dx \overset{\text{def}}{=} \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 8.4

Evaluate the definite integral

$$\int_0^2 x \ln(x)\,dx.$$

Solution.

$$\begin{aligned}
&\int_0^2 x \ln(x)\,dx \\
=&\lim_{t \searrow 0} \int_t^2 x \ln(x)\,dx \\
=&\lim_{t \searrow 0} \left(\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2\right)\Big|_t^2 \\
=&\lim_{t \searrow 0} \left(2\ln(2) - 1 - \frac{1}{2}t^2 \ln(t) + \frac{1}{4}t^2\right) \\
=&2\ln(2) - 1 - \frac{1}{2}\lim_{t \searrow 0}(t^2 \ln(t)) + \frac{1}{4}\lim_{t \searrow 0} t^2 \\
=&2\ln 2 - 1
\end{aligned}$$

Because we have $\lim_{t \searrow 0} t^2 = 0$ and

$$\lim_{t \searrow 0} t^2 \ln(t) = \lim_{t \searrow 0} \frac{\ln(t)}{\frac{1}{t^2}}$$

and by L'Hopital's Rule,

$$\lim_{t \searrow 0} \frac{\frac{1}{t}}{-2\frac{1}{t^3}} = \lim_{t \searrow 0}\left(-\frac{1}{2}t^2\right) = 0. \quad \square$$

Theorem 8.5 (Inequality Comparison)

Let $f, g : [a, +\infty) \to \mathbb{R}$ be continuous functions. Suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges
2. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges, and $0 \leq \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$

Example 8.6

Determine whether $\int \frac{\sin(x^2)}{x^2}\,dx$ diverges or converges.

Solution. Notice that $0 \leq \sin^2(x^2) \leq 1$, so $\frac{\sin^2(x^2)}{x^2} \leq \frac{1}{x^2}$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty} \left(-\frac{1}{x}\right)_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right)$$
$$=1$$

So the function converges at a value less than 1. □

§8.1 Asymptote Notation

Definition 8.7. Let f and g be non-negative continuous functions.
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then we write $f \sim g$ and say f is **asymptotic** to g.
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then we write $f = o(g)$ and say f is **little o** of g.

Theorem 8.8 (Limit Comparison)

Suppose $f \sim c \cdot g$ for some constant $c > 0$. Then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.
If $f = o(g)$, then:

1. If f diverges, then g diverges
2. If g converges, then f converges.

Example 8.9

Find

$$\lim_{x\to\infty} \frac{\sqrt{6x^7-5x-1}+\ln(x)}{x^5+\cos(x^3)}$$

Solution. $\ln(x)+\sqrt{6x^7-5x-1} \sim \sqrt{6x^7} = \sqrt{6}x^{\frac{7}{2}}$ and $x^5+\cos(x^3) \sim x^5$

$$\implies \lim_{x\to\infty} \frac{\sqrt{6x^7-5x-1}+\ln(x)}{x^5+\cos(x^3)} \sim \sqrt{6}\frac{x^{\frac{7}{2}}}{x^5} = \sqrt{6}x^{-\frac{3}{2}}$$

□

§9 September 12, 2018

§9.1 Sequences and Series

Sequences

Definition 9.1. A **sequence** is an *ordered* list of numbers $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$, which can also be written as $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$.

Example 9.2 (Some Motivation for Sequences)
Some examples for which sequences are used

1. **Discrete sampling** of (continuous) functions with $a_n = f(n)$ for all positive integers n. Sequence given by an explicit formula e.g. $a_n = 2^n$. This makes sense in cases when it is only possible or realistic to get the value of f at certain time intervals.

2. **Recurrence relation**: $a_{n+1} = f(a_1, a_2, \ldots, a_n)$. e.g. Fibonacci sequence
Motivation: discrete dynamical system
Another example of this is Taylor series, which defines a sequence $a_0, a_1, \ldots$ so that $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ for x near 0. The sequence satisfies that the first n terms represents the best $n-1$ degree polynomial approximation for a given function.

Now suppose we have a sequence $a_1, a_2, \ldots$. What should "$\frac{d}{dn} a_n$" or "$\int a_n$" mean?

Intuition tells us that the derivatives should be $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$ and that integrals should be $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$. Indeed, we can check that "taking the derivative" of the integrals will yield the original sequence again, and "taking the integral" of the derivatives also yields the original sequence, so our intuition seems to be right.

It turns out that this is precisely the definition of derivatives and integrals, which is encapsulated in the Fundamental Theorem of Calculus for sequences. This is because while Riemann sums for continuous functions get more refined as the change we make to the variable smaller, we have already made the smallest possible change in the sequences.

§9.2 Limits of Sequences

We first formalize what the limit of a sequence is as the sequence becomes infinitely long.

Definition 9.3. $\lim_{n\to\infty} a_n = L$ means given any error bound $\epsilon > 0$, we can find some N such that $|a_n - L| \leq \epsilon$ whenever $n \geq N$.

Example 9.4

$$\lim_{n\to\infty} \frac{1}{\log(n)} = 0$$

because for any $\epsilon > 0$ and $n > 10^{\frac{1}{\epsilon}}$, we have

$$\left|\frac{1}{\log(n)}\right| < \frac{1}{\log(10^{\frac{1}{\epsilon}})} = \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

Theorem 9.5 (Limit of Sequence is Limit of Function)
Suppose f is a function on $\mathbb{R}$ and define $a_n = f(n)$ for all positive integers n. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Proposition 9.6 (Limit Laws)
$\lim_{n\to\infty} c = c$
$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

Generally, if f is continuous, then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$.

Now we are ready to talk about the convergence of sequences.

Convergence of Sequences

Definition 9.7. We say that $\{a_n\}$ is **bounded** if there exists B such that $|a_n| \le B$ for all n.

Definition 9.8. We say that $\{a_n\}$ is **monotone** if either

- $n \le m \implies a_n \le a_m$, or
- $n \le m \implies a_n \ge a_m$.

Theorem 9.9 (Monotone Convergence)
If $\{a_n\}$ is bounded and monotone, then $\lim_{n\to\infty} a_n$ exists.

The following theorem is an analogue of the Squeeze Theorem for continuous functions, and the proof for functions also works for sequences.

Theorem 9.10 (Squeeze Theorem for Sequences)
If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} c_n$ exist and are equal, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

§10 September 14, 2018

Example 10.1

Determine whether the following sequences converge:

1. $a_n = \dfrac{n(n-2)(n-4)}{3n^3}$
2. $b_n = \cos(n) + \dfrac{\ln(n)}{n^2}$
3. $c_n = \dfrac{\cos(n) - \ln(n)}{\sqrt{n}}$

Solution. a_n **converges**, because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^3 - 6n^2 + 8n}{3n^3} = \lim_{n\to\infty} \dfrac{1 - \frac{6}{n} + \frac{8}{n^2}}{3} = \dfrac{1}{3}$ since $\lim_{n\to\infty} \dfrac{6}{n} = 0$ and $\lim_{n\to\infty} \dfrac{8}{n^2} = 0$.
b_n **diverges** because $\dfrac{\ln(n)}{n}$ converges (due to $\ln(n) = o(n^2)$) but $\cos(n)$ oscillates between -1 and 1.
c_n **diverges** also because $\cos(n)$ is bounded while $\sqrt{n}$ increases as n approaches ∞, so that $\dfrac{\cos(n)}{\sqrt{n}}$ converges to 0, and $\ln(n) = o(n) \implies \dfrac{\ln(n)}{\sqrt{n}}$ converges to 0. □

Now that we have talked about sequences, we can talk about the concept of series.

Series

Given a sequence $a_1, a_2, a_3, \ldots$, define the sequence $S_1, S_2, S_3, \ldots$ by

$$S_k = a_1 + a_2 + \ldots + a_k.$$

This is very cumbersome to write individually, so we use the notation $\sum_{i=1}^{k} a_i$ to denote the sum of a_i from $i = 1$ to k. Then we have the shorthand notation $S_k = \sum_{i=1}^{k} a_i$.

Definition 10.2. Define the infinite series

$$\sum_{i=1}^{\infty} a_i = \lim_{k\to\infty} \sum_{i=1}^{k} a_i = \lim_{k\to\infty} S_k.$$

These are some common types of sequences.

1. **Arithmetic Sequences:** $a_0 = a, a_{n+1} = a_n + b, a_n = a + n \cdot b$.
2. **Geometric Sequences:** $a_0 = a, a_{n+1} = a_n n \cdot r, a_n = ar^n$.

In particular, the geometric sequence has a lot of applications in science.
The **geometric series** is given by

$$S_k = a + ar + ar^2 + \cdots + ar^k = a(1 + r + r^2 + \ldots + r^k) = a \cdot \frac{1 - r^{k+1}}{1 - r} \text{ for } r \neq 1.$$

This formula holds because

$$(1-r)(1+r+r^2+\ldots+r^k) = 1-r+r-r^2+r^2-r^3+\ldots-r^{k+1} = 1-r^{k+1}.$$

So $\sum_{n=0}^{\infty} ar^i = \lim_{k\to\infty} S_k = \frac{a}{1-r}$ if $|r|<1$, but the sum diverges if $|n| \geq 1$.

§10.1 Telescoping Sums

The main idea behind telescoping sums is breaking down the value into a positive and negative component, typically by using partial fractions, so that terms cancel out. "Telescoping" means that because stuff in the middle cancel out, we're only really looking at the front and end of the sequence and ignoring the middle (like a telescope!)

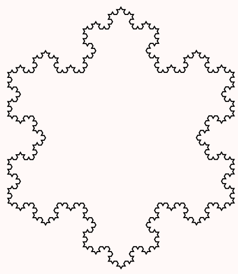
Example 10.3

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \\
&= \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1
\end{aligned}$$

Okay, for some reason, we discussed that really briefly and jumped right back into what we were previously discussing.

Example 10.4 (Koch Snowflake)

The Koch snowflake is a fractal constructed by starting with an equilateral triangle, and then recursively adding new parts by dividing each segment into three, drawing an equilateral triangle with the middle segment as the base, and then removing that base segment.



Find the perimeter and area of the infinite Koch snowflake.

Solution. We can try to obtain the perimeter P_n of the Koch snowflake at step n.
$P_1 = 3, P_2 = 3 \cdot \left(\frac{4}{3}\right), P_3 = 3\left(\frac{4}{3}\right)^2, P_n = 3\left(\frac{4}{3}\right)^n.$
We note that the perimeter is a geometric sequence, and since the common ratio is ≥ 1, the sequence diverges as $n \to \infty$.

If we let A_n, E_n be the area and number of edges of the figure at step n, respectively, and T_n be the area of each triangle at step n, then $E_0 = 3, E_1 = 3 \cdot 4, \ldots, E_n = 3 \cdot 4^n$. Then $T_0 = \frac{\sqrt{3}}{4}, T_1 = \frac{\sqrt{3}}{4} \cdot \frac{1}{9}, T_2 = \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^2, \ldots, T_n = \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^2$.

$$\begin{aligned}
A_0 &= \frac{\sqrt{3}}{4} \\
A_n &= A_{n-1} + E_{n-1} \cdot T_n \\
&= A_{n-1} + 3 \cdot 4^{n-1} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^n \\
&= A_{n-1} + \frac{\sqrt{3}}{4} \cdot \frac{3}{4} \left(\frac{4}{9}\right)^n \\
&= \frac{\sqrt{3}}{4} \left[1 + \frac{3}{4} \left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \ldots + \left(\frac{4}{9}\right)^n\right)\right] \\
&= \frac{\sqrt{3}}{4} \left[1 + \frac{3}{4} \left(\frac{4}{9} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}}\right)\right] \\
&= \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \left(\frac{1 - \left(\frac{4}{9}\right)^n}{\frac{5}{9}}\right)\right] \\
&= \frac{\sqrt{3}}{4} \left[1 + \frac{3}{5} \left(1 - \left(\frac{4}{9}\right)^n\right)\right] \\
\lim_{n\to\infty} A_n &= \frac{\sqrt{3}}{4} \left(1 + \frac{3}{5}\right) = \frac{\sqrt{3}}{4} \cdot \frac{8}{5}
\end{aligned}$$

□

Now we talk about an important theorem that will probably simplify our work a lot in tests.

It is intuitive to think that a a series will only converge if eventually, the terms of the sequence are small enough that adding them to the series will only cause the series to approach some value L but not reach it. Let us formalize this statement mathematically.

Suppose $\sum_{n=1}^{\infty} a_n = L$ and let $S_k = \sum_{n=1}^{k} a_n$. Then $a_k = S_k - S_{k-1}$.
So $\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = L - L = 0$ (i.e. the sequence converges to 0). It turns out that this is the statement of the the Divergence Theorem!

Theorem 10.5 (Divergence Theorem)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

We have to be careful here. The Divergence Theorem tells us that if the series converges, then the sequence[3] must converge to 0, and if the sequence doesn't converge to 0 (i.e.

[3] Remember that the sequence is just a bunch of numbers, while the series is the sum of the first n terms of the sequence.

converges to a non-zero value or diverges), then the series diverges. However, it doesn't tell us anything about the series if the sequence converges to 0. For example, take the harmonic sequence below.

> **Example 10.6** (A Non-Example)
> The harmonic series given by
> $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$
> diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

If this is still confusing, just remember that we can only use the Divergence Theorem to show that a **sequence converges to 0** or to show that a **series diverges**.

> **Example 10.7** (Divergence Theorem in Action)
> We present more examples of when Divergence Theorem can be applied:
>
> - $\sum_{n=1}^{\infty} e^{\frac{1}{n^2}}$ diverges because $\lim_{n\to\infty} e^{\frac{1}{n^2}} = e^0 = 1$.
> - $\sum_{n=1}^{\infty} \frac{1}{n^3}$ cannot be determined to converge or diverge by the Divergence Theorem because while $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the Divergence Theorem can't tell us anything about the series.
> - $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3n-1}$ diverges because $\lim_{n\to\infty} \frac{(-1)^n \cdot n}{3n-1} = (-1)^n \cdot \frac{1}{3}$, which diverges.

§11 September 19, 2018

We go over some examples of types of problems that were glossed over in previous classes.

> **Example 11.1**
> Does $\sum_{n=3}^{\infty}(e^{-n^2} - e^{-(n+1)^2})$ converge? If so, to what value?

Solution. Note that the first few values of the series is given by $\frac{1}{e^9} - \frac{1}{e^{16}} + \frac{1}{e^{16}} - \frac{1}{e^{25}} + \frac{1}{e^{25}} - \frac{1}{e^{36}} + \dots$, which is a telescoping sum!

Then $\sum_{n=3}^{\infty}(e^{-n^2} - e^{-(n+1)^2}) = \frac{1}{e^9} + (\text{middle terms that cancel out}) + \lim_{k\to\infty} \frac{1}{e^{(k+1)^2}} = \frac{1}{e^9}$.[4]

□

[4] Remember to write this a little more formally by taking $\lim_{k\to\infty}$ of the sum during an actual test.

Example 11.2

Let $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}, a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, and in general, $a_{n+1} = \sqrt{2a_n}$. Does $\{a_n\}$ converge? If so, to what?
Hint: Is $\{a_n\}$ bounded? Is it monotone?

Solution. Claim: $\sqrt{2a_n} > a_n$.
If $\sqrt{2x} > x$ and $x > 0$, then $2x > x^2 \implies 0 < x < 2$. Indeed, if $a_n \in (0, 2)$, then $a_n < \sqrt{2a_n}$ and $0 < \sqrt{2a_n} < 2$, which means that $\{a_i\}$ is monotonically increasing and bounded. So $\lim_{n\to\infty} a_n$ converges to some limit L. But we also have that $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2L}$, so $L^2 = 2L \implies L = 0$ or $L = 2$. Since clearly $L \neq 0$, we must have $L = 2$.

An easier way to solve this[5] is to notice that $a_1 = 2^{\frac{1}{2}}, a_2 = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{2}+\frac{1}{4}}, \ldots$ so that the general $a_n = 2^{\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}} = 2^{1-\frac{1}{2^n}}$ which approaches 2 as $n \to \infty$. □

Example 11.3

Let $a_1 = C, a_{n+1} = a_n^2$. For which C does $\{a_n\}$ converge?

Solution. If $\lim_{n\to\infty} a_n = L$, then $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n^2 = L^2$, so $L = 0$ or $L = 1$. If $C \in (-1, 1)$, the sequence converges to 0, and if $C = -1, 1$, then the sequence converges to 1. □

§12 September 21, 2018

§12.1 Review of Integration Techniques

Exercise 12.1. $\displaystyle\int \tan^2 x \sec x \, dx$

Solution.

$$\begin{aligned}
&\int \tan^2 x \sec x \, dx \\
=& \int (\sec^2 x - 1) \sec x \, dx \\
=& \int \sec^3 x \, dx - \int \sec x \, dx \\
=& \sec x \tan x - \int \tan^2 x \sec x \, dx - \int \sec x \, dx
\end{aligned}$$

where the last line follows from integration by parts (i.e. $u = \sec x \implies du = \sec x \tan x \, dx$ and $v = \tan x \implies dv = \sec^2 x \, dx$). So if we let the integral be B, we have

$$B = \int \sec^2 x \, dx - \int \sec x \, dx = \sec x \tan x - B - \int \sec x \, dx$$

$$\implies 2B = \sec x \tan x - \int \sec x \, dx \implies B = \frac{1}{2}(\sec x \tan x - \int \sec x \, dx)$$

□

[5] This was communicated to the class by Anastasia Romanova

Exercise 12.2. $\cos\theta(1-\cos\theta)^{2018}\sin\theta\,d\theta$

Solution. The problem seems to be easier to manipulate if we use u-substitution.

$$\begin{aligned}
&\cos\theta(1-\cos\theta)^{2018}\sin\theta\,d\theta \\
=&\int(1-u)u^{2018}\,du \\
=&\int u^{2018}-u^{2019}\,du \\
=&\frac{1}{2019}u^{2019}-\frac{1}{2020}u^{2020}+C \\
=&\frac{1}{2019}(1-\cos\theta)^{2019}-\frac{1}{2020}(1-\cos\theta)^{2020}+C.
\end{aligned}$$

□

Exercise 12.3. $\displaystyle\int\frac{x^2-2}{x(x-1)(x-2)}\,dx$

Exercise 12.4. $\displaystyle\int(x^6-x^4)^{-\frac{1}{2}}\,dx$

Solution. We use trig substitution with $x=\sec\theta \implies dx=\sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}
&\int(x^6-x^4)^{-\frac{1}{2}}\,dx \\
=&\int\frac{1}{x^2\sqrt{x^2-1}}\,dx \\
=&\int\frac{1}{\sec^2\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta \\
=&\int\frac{1}{\sec^2\theta\tan\theta}\sec\theta\tan\theta\,d\theta \\
=&\int\cos\theta\,d\theta \\
=&\sin\theta+C \\
=&\sin\sec^{-1}x+C.
\end{aligned}$$

□

Exercise 12.5. $\displaystyle\int\cos\ln 2x\,dx$

Solution. We use u-substitution with $u=\ln(2x)\implies du=\frac{1}{x}\,dx$ and $e^u=2x\implies x=\frac{e^u}{2}$.

$$\begin{aligned}
&\int\cos\ln 2x\,dx \\
=&\int\frac{x\cos\ln 2x}{x}\,dx \\
=&\int\frac{e^u}{2}\cos u\,du \\
=&\frac{e^u}{2}\cos u+\int\frac{e^u}{2}\sin u\,du \\
=&\frac{e^u}{2}\cos u+\frac{e^u}{2}\sin u-\int\frac{e^u}{2}\cos u\,du
\end{aligned}$$

Then letting $A = \int \frac{e^u}{2} \cos u \, du$, we have $A = \frac{e^u}{2} \cos u + \frac{e^u}{2} \sin u - A$
$\implies A = \frac{e^u}{4} \cos u + \frac{e^u}{4} \sin u = \frac{1}{2} x \cos \ln 2x + \frac{1}{2} x \sin \ln 2x.$ □

§13 September 24, 2018

§13.1 Review of Definitions

Improper Integrals

- $\int_a^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx.$
- $\int_{-\infty}^{a} f(x) \, dx = \lim_{t \to -\infty} \int_t^a f(x) \, dx.$
- If f is defined on $[a, b)$:
  $\int_a^b f(x) \, dx = \lim_{t \to b} \int_a^t f(x) \, dx.$

Sequences and series

- A sequence is an infinite list of numbers.
- Given a sequence $\{a_n\}_{n=1}^{\infty}$, we have a sequence of partial sums $S_n = \sum_{i=1}^{n} a_i = a_1 + \ldots + a_n$.
- Define $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_i$.
- Ex: Geometric series: $a_1 = a, a_{n+1} = r \cdot a_n \implies a_n = ar^n$.
  $\sum_{i=0}^{n} a_i = a + ar + \ldots + ar^n = a \cdot \frac{1 - r^{n+1}}{1 - r}$.
  If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$.

Telescoping Series

- $\sum_{n=5}^{\infty} \frac{1}{n(n+1)} = \sum_{n=5}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \ldots$
  $= \lim_{k \to \infty} \sum_{n=5}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{k \to \infty} \left(\frac{1}{5} - \frac{1}{k+1} \right) = \frac{1}{5}.$
- The following is an example of a telescoping series that does not converge:
  $\sum_{n=1}^{\infty} (\log(n) - \log(n+1)) = \lim_{k \to \infty} \sum_{n=1}^{k} (\log(n) - \log(n+1))$
  $= \lim_{k \to \infty} (\log(1) - \log(k+1)) = \infty$

Monotone convergence

- A sequence $\{a_n\}$ is bounded if there exists B such that $|a_n| \leq B$ for all n.

- $\{a_n\}$ is monotone if either $n \leq m \implies a_n \leq a_m$ or $n \leq m \implies a_n \geq a_m$.
- Monotone Convergence Theorem: If $\{a_n\}$ is bounded and monotone, then $\{a_n\}$ converges.

Inequality comparison tests
Suppose $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

- If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, and $0 \leq \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$.

Suppose $0 \leq a_n \leq b_n$ for all n.

- If $\sum_{n=1}^\infty b_n$ converges, then so does $\sum_{n=1}^\infty a_n$, and $0 \leq \sum_{n=1}^\infty a_n \leq \sum_{n=1}^\infty b_n$

Asymptote Notation
Let $f, g \geq 0$.

- $f \sim g \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$
- $f = o(g) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
  Limit Comparison Theorem:
- If $f \sim g$, then $\int_a^\infty f(x)\,dx$ converges $\iff \int_a^\infty g(x)\,dx$ converges.
- If $f = o(g)$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

Exercises

Exercise 13.1. What is $\frac{(x^4+5x-\log(x))^{\frac{1}{3}}}{\sin^2 x + x^2 + e^x}$ asymptotic to?

Solution. $\frac{(x^4+5x-\log(x))^{\frac{1}{3}}}{\sin^2 x + x^2 + e^x} \sim \frac{x^{\frac{4}{3}}}{x^2}$ □

§14 September 27, 2018

§14.1 Integral Test

Let f be a continuous decreasing non-negative function. For $n \geq b$, let $a_n = f(n)$. Our goal is to estimate $\sum_{n=b+1}^{\infty} a_n$. We have that

$$\int_{b+1}^{M+1} f(x)\,dx < \sum_{n=b+1}^{M} a_n < \int_b^M f(x)\,dx.$$

If we take the limit of the inequality as $M \to \infty$, we obtain the following:

Theorem 14.1 (Integral Test)

For a nonnegative, continuous, decreasing function $f(n)$ and a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq b$,

1. $\displaystyle\sum_{n=b+1}^{\infty} a_n$ converges $\iff \displaystyle\int_{b+1}^{\infty} f(x)\,dx$ converges.
2. If they both converge, then

$$\int_{b+1}^{\infty} f(x)\,dx < \sum_{n=b+1}^{\infty} a_n < \int_{b}^{\infty} f(x)\,dx.$$

§14.2 p-series

Definition 14.2. Let p be a real number. Then the ***p*-series** is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Claim 14.3. The p-series converges for $p > 1$ and diverges for $p \leq 1$.

Proof. If $p \leq 0$, then $\dfrac{1}{n^p} \not\to 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the divergence test.
If $p = 1$, then the p-series is just the harmonic series, which diverges.
If $0 < p < 1$, then the series diverges by Inequality Comparison with the harmonic series.
If $p > 1$, then the series converges by the Integral Test, since $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ converges. □

We can reword this under the following proposition.

Proposition 14.4

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff \int_1^{\infty} x^{-p}\,dx \text{ converges.}$$

Proof. $\displaystyle\int_1^{\infty} x^{-p}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\,dx$. For $p \neq 1$, this is equivalent to $\displaystyle\lim_{t\to\infty} \left(\frac{1}{-p+1} x^{-p+1}\right)_1^t =$ $\displaystyle\lim_{t\to\infty} \left(\frac{1}{-p+1} t^{-p+1} - \frac{1}{-p+1}\right) = \frac{1}{p-1}$ for all $p > 1$. □

So for $p > 1$:

$$1 + \frac{1}{p-1} \cdot 2^{1-p} = 1 + \int_2^{\infty} \frac{1}{x^p}\,dx < 1 + \sum_{n=2}^{\infty} \frac{1}{n^p} < 1 + \int_1^{\infty} \frac{1}{x^p}\,dx = 1 + \frac{1}{p-1}$$

In particular, for $p = 2$, we get $\displaystyle\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$. In fact, the actual value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\dfrac{\pi^2}{6}$.

§14.3 Riemann Zeta Function

Definition 14.5. For $s > 1$, denote $\zeta(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$.

For example, $\frac{1}{\zeta(2)}$ is the probability that 2 random integers are coprime.

§15 September 28, 2018

Example 15.1

How fast does $\sum_{n=3}^{M} \frac{1}{n \ln n}$ grow as $M \to \infty$?

Solution. By the integral test,

$$\int_3^{M+1} \frac{1}{x \ln x}\, dx < \sum_{n=3}^{M} \frac{1}{n \ln x} < \int_2^{M} \frac{1}{x \ln x}\, dx.$$

We have that $\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C$ for $u = \ln x$, i.e. $\int \frac{1}{x \ln x}\, dx = \ln|\ln x| + C$. So $\int_a^b \frac{1}{x \ln x} = \ln|\ln x|_a^b$. So

$$\ln|\ln(M+1)| - \ln|\ln 3| < \sum_{n=3}^{M} \frac{1}{n \ln n} < \ln|\ln M| - \ln|\ln 2|$$

$$\implies \left| \sum_{n=3}^{M} \frac{1}{n \ln n} - \ln \ln M \right| \text{ is bounded.}$$

□

Theorem 15.2 (An Analog of the Integral Test for Increasing Functions)

For a nonnegative, continuous, increasing function f such that $f(n) = a_n$ for all $n \geq b$,

$$\int_b^M f(x)\, dx \leq \sum_{n=b+1}^{M} a_n \leq \int_{b+1}^{M+1} f(x)\, dx.$$

Example 15.3

Estimate the value of $\sum_{n=1}^{M} n^2$ as $M \to \infty$.

Solution. By the integral test,

$$\frac{1}{3}M^3 = \int_0^M x^2\, dx \leq \sum_{n=1}^{M} n^2 \leq \int_1^{M+1} x^2\, dx = \frac{1}{3}(M+1)^3 - \frac{1}{3} = \frac{1}{3}M^3 + M^2 + M.$$

This is pretty close, as the actual value

$$\sum_{n=1}^{M} n^2 = \frac{M(M+1)(2M+1)}{6} = \frac{1}{3}M^3 + \frac{1}{2}M^2 + \frac{1}{6}M.$$

□

Below we will give the analogs of some theorems and definitions in integrals for series.

Theorem 15.4 (Inequality Comparison for Series)
Suppose $0 \le a_n \le b_n$.

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=0}^{\infty} b_n$.

Suppose $a_n \ge 0, b_n \ge 0$ for all n.

Definition 15.5. $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Definition 15.6. $a_n = o(b_n)$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Theorem 15.7 (Limit Comparison for Series)
Let $\{a_n\}, \{b_n\}$ be sequences.

1. If $a_n \sim b_n$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
2. If $a_n = o(b_n)$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

§16 October 1, 2018

Example 16.1 (Warmup Problem)
Do the following series converge?

1. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$
2. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$

Solution. Let's compare the first series with the harmonic series. We have $\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} n^{-\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(n^{-\frac{1}{n}})} = \lim_{n\to\infty} e^{-\frac{1}{n}\ln(n)} = e^0 = 1$. This means that the series is asymptotic to the harmonic series. Since the harmonic series diverges, the series also diverges. □

The second series converges. We have that $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \sum_{n=1}^{\infty} \frac{1}{(n+1)(\sqrt{n+1}+\sqrt{n+1})} = \sum_{n=1}^{\infty} \frac{1}{2(n+1)^{\frac{3}{2}}} = \sum_{n=2}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$. Because the p-series with $p = \frac{3}{2}$ converges, the series also converges. □

Example 16.2

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

§16.1 Alternating Series

We know that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. But what about $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$?

Well we know that $\{S_{2k}\}$ is bounded and increasing, i.e. $S_2 \le S_4 \le S_6 \le \ldots \le 1$, while $\{S_{2k+1}\}$ is bounded and decreasing, i.e. $S_1 \ge S_3 \ge S_5 \ge \ldots \ge 0$.

We actually have that $S = S_{odd} = S_{even}$. Indeed, if we let $S_{odd} = \lim_{k\to\infty} S_{2k+1}$ and $S_{even} = \lim_{k\to\infty} S_{2k}$, then we have

$$\begin{aligned} S_{odd} - S_{even} &= \lim_{k\to\infty} S_{2k+1} - \lim_{k\to\infty} S_{2k} \\ &= \lim_{k\to\infty} (S_{2k+1} - S_{2k}) \\ &= \lim_{k\to\infty} (-1)^{2k+2} \frac{1}{2k+1} \\ &= \lim_{k\to\infty} \frac{1}{2k+1} = 0. \end{aligned}$$

Theorem 16.3 (Alternating Series Test)

Suppose $0 \le b_{n+1} \le b_n$ for all $n \ge 1$, and $\lim_{k\to\infty} b_k = 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ and $\sum_{n=1}^{\infty} (-1)^n b_n$ both converge.

The Alternating Series Test just tells us that if we have a monotone decreasing sequence whose limit approaches 0, then the alternating series taken from that series will also converge. Thus, we can estimate the value of the alternating series using only the first few terms.

It is natural to ask what the error would be if we only take the sum of the first few terms, i.e. get the remainder estimate.[6] We have that if $R_M = S - S_M = \sum_{n=M+1}^{\infty} (-1)^{n+1} b_n$, then

$$|R_M| \leq |S_{M+1} - S_M| = b_{M+1}.$$

Example 16.4

Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. How big does M have to be for $|R_M| \leq \frac{1}{10000}$?

Solution. We have that $b_n = \frac{1}{n^2}$, and we know that $\lim_{n\to\infty} b_n = 0$, so the alternating series indeed converges. Then $|R_M| \leq \frac{1}{10000}$ holds for $M \geq \sqrt{10000} - 1 = 99$, i.e. $|R_{99}| \leq b_{100} = \frac{1}{10000}$. □

Definition 16.5. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.

Definition 16.6. $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

§17 October 3, 2018

Theorem 17.1

Absolute convergence implies convergence.

Proof. Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We have

$$|a_n| + a_n = \begin{cases} 2a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$

so $0 \leq |a_n| + a_n \leq 2|a_n|$, so $\sum_{n=1}^{\infty} (|a_n| + a_n)$ converges by inequality comparison. Thus, $\sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$ converges. □

Remark 17.2. If $\sum_{n=1}^{\infty} a_n$ converges *absolutely*, then we can reorder the terms freely. But if $\sum_{n=1}^{\infty} a_n$ converges *conditionally*, reordering terms may make it diverge or change in value.

[6] This is similar to what we did for the Integral Test

Example 17.3 (The Alternating Harmonic Series)

The alternating harmonic series is given by $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$.

Suppose we rearrange this series as $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$, then the new sum is no longer $\ln 2$ but $\frac{3}{2} \ln 2$.

If we rearrange the series again into

$$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k+1}}_{\geq 100} - \frac{1}{2} + \underbrace{\frac{1}{2k+3} + \ldots + \frac{1}{2j+1}}_{\geq 1000} - \frac{1}{4} + \ldots$$

then the sum changes even more!

A way to think of this seemingly counterintuitive phenomenon is that the sum of an alternating series is a tug of war between the convergence of the sum of positive terms and the convergence of the sum of the negative terms. Moving infinitely many positive terms in front will then shift this tug of war to the side of positive even more.

Theorem 17.4 (Ratio Test)

Let $a_1, a_2, a_3, \ldots$ be nonzero and $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

(i) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\rho = 1$, the ratio test gives no information.

An example of this is that for the harmonic series, $\rho = 1$ and the series diverges, but for the alternating harmonic series, $\rho = 1$ too but the series converges.

Example 17.5

Use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

Solution. $\rho = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \left(\frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \right) = \lim_{n\to\infty} \frac{2}{n+1} = 0$, so $\sum_{n=1}^{\infty} \frac{2}{n!}$ converges absolutely. □

Example 17.6

Does $\sum_{n=1}^{\infty} \frac{n^{2018}}{3^n}$ converge absolutely, converge conditionally, or diverge?

Solution. The series converges absolutely. We have that $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^{2018} \cdot 3^{-n-1}}{n^{2018} \cdot 3^{-n}} \right| = \lim_{n\to\infty} \frac{1}{3} \left| \frac{(n+1)^{2018}}{n^{2018}} \right| = \lim_{n\to\infty} \frac{1}{3} \left| \left(\frac{n+1}{n} \right)^{2018} \right| = \frac{1}{3}$, so the series converges absolutely. □

Theorem 17.7 (Root Test)

Let $a_1, a_2, a_3, \ldots$ be nonzero and $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$.

(i) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\rho = 1$, the ratio test gives no information.

Notice that the Root Test is just the same statement as the Ratio Test, except for its condition of what ρ is.

Strategies for Proving Convergence

- Divergence Test
- Geometric Series
- p-series
- Integral Test (with remainder estimate)
- Inequality/limit comparison
- Alternating series test
- Ratio test
- Root test

§18 October 5, 2018

Example 18.1

Determine whether the series converges absolutely, converges conditionally, or diverges:

1. $\sum_{n=1}^{\infty} \frac{-n+1}{n^3+n^2+n+1}$
2. $\sum_{n=57}^{\infty} \frac{(n+1)^2}{n^3+(1.1)^n}$

Solution. See following:

1. We use the limit comparison on the p-series for $p = 2$. $\lim_{n\to\infty} \frac{\frac{-n+1}{n^3+n^2+n+1}}{\frac{1}{n^2}} = -1$. Note however that the limit comparison theorem only works for *positive* terms, so we factor out the -1, and so $-\lim_{n\to\infty} \frac{\frac{n-1}{n^3+n^2+n+1}}{\frac{1}{n^2}} = +1$, so the series **converges absolutely** by limit comparison with the p-series.

2. We use the ratio test: $\lim_{n\to\infty} \frac{(n+2)^2}{(n+1)^3+(1.1)^{n+1}} \cdot \frac{n^3+(1.1)^n}{(n+1)^2} = \frac{1}{1.1} < 1$, since $\frac{n^3+(1.1)^n}{(n+1)^3+(1.1)^{n+1}} = \frac{n^3+(1.1)^n}{(n+1)^3+(1.1)^{n+1}} \cdot \frac{(1.1)^{-n}}{(1.1)^{-n}} = \frac{1+\frac{n^3}{(1.1)^n}}{1.1+\frac{(n+1)^3}{(1.1)^n}} = \frac{1}{1.1}$. Thus, the series converges absolutely.[7] □

Example 18.2

For which real numbers x does these series converge?

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
3. $\sum_{n=1}^{\infty} \frac{2^n}{n^x}$
4. $\sum_{n=0}^{\infty} (1-x)^n$

Solution. See following:

1. Ratio test: $\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} = \frac{x}{n+1}$, and $\lim_{n\to\infty} \frac{x}{n+1} = 0$, so the series converges absolutely for all x.[8]

2. We use the ratio test: $\left| \frac{(-1)^{n+2} \cdot \frac{x^{n+1}}{n+1}}{(-1)^{n+1} \frac{x^n}{n}} \right| = \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = \frac{n}{n+1} \cdot |x|$. So $\rho = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x|$. So $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ converges absolutely for $-1 < x < 1$, diverges for $x > 1$ or $x < 1$. For $x = 1$, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is just the alternating harmonic series, so the series converges conditionally. Similarly, for $x = -1$, the

[7] It doesn't matter that n starts at 57. In fact, n can start at 1 or 200 or 1000000 and the series will still converge. This is because adding or subtracting a finite value will not change the convergence of a series.

[8] This series actually converges to e^x and is called the Taylor Series of e.

series is the negative alternating harmonic series, which converges conditionally too.[9]

3. We can use the root test for this problem. $\rho = \lim_{n\to\infty} \sqrt[n]{|1-x|^n} = |1-x|$, so the the series converges absolutely, for $0 < x < 2$, diverges for $x < 0$ or $x > 2$. For $x = 2$, $\sum_{n=0}^{\infty}(-1)^n$ diverges. For $x = 0$, $\sum_{n=0}^{\infty}(1-0)^n = \sum_{n=0}^{\infty} 1$ diverges. □

§18.1 Taylor Series

Main Idea: We want to approximate functions by polynomials.[10]

Motivation: We start with the linear case. For a function $y = f(x)$ and a point on the function $(a, f(a))$, the best linear approximation for x near a can be approximated by $f(x) \approx f(a) + f'(a)\cdot(x-a)$. But obviously, this can be better; i.e. we can do better with a quadratic!

$f(x) \approx f(a) + f(a)\cdot(x-a) + c_2(x_a)^2$, and we have that $f'(x) \approx f'(a) + f''(a)\cdot(x-a) = f'(a) + 2c_2(x-a)^2$, so we get that $c_2 \approx \frac{1}{2}f''(a)$. Thus, the best quadratic approximation is $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

§19 October 10, 2018

Let f be a smooth function and $x_0 \in \mathbb{R}$. Then the linear approximation for f is given by

$$f(x) \approx P_1(x) = f(x_0) + f'(x_0)\cdot(x-x_0).$$

$P_1(x_0) = f(x_0)$
$P_1'(x_0) = f'(x_0)$
We want $P_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots + c_n(x-x_o)^n$ such that

$$P_n^{(k)}(x_0) = f^{(k)}(x_0) \text{ for each } k = 0, 1, \ldots, n$$

such that this is the best degree $\le n$ polynomial approximation of f near x_0. Then
$P_n'(x) = f'(x_0) + 2c_2(x-x_0) + 3c_3(x-x_0)^2 + \ldots + nx_n(x-x_0)^{n-1}$
$P''(n) = \underbrace{2c_2}_{\text{need } c_2 = \frac{1}{2}f''(x_0)} + 3\cdot 2c_3(x-x_0) + \ldots + n(n-1)c_n(x-x_0)^{n-2}$
Continuing like this, we get $k!\cdot c_k = f^{(k)}(x_0)$. So

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \ldots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n.$$

Definition 19.1. P_n is the n^{th} **Taylor polynomial** of f at x_0.

Definition 19.2. If $x_0 = 0$, this is called the **Maclaurin polynomial**.

Example 19.3
Let $f(x) = c_0 + c_1x + \ldots + c_Mx^M$. What is the n^{th} Maclaurin polynomial of f.

[9] This series is actually the Taylor Series for the natural logarithm ln function.
[10] Because polynomials are the easiest non-constant polynomials that can be evaluated

Solution. If $n \geq M$, then $P_n(x) = f(x)$ because this function is exactly the one we want to approximate.
Now suppose $n \leq M$, let's try out some values:
$P_2(x) = c_0 + c_1x + c_2x^2$, and in general,
$P_n(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$. ☐

Remark 19.4. For approximating polynomials, the n^{th} Maclaurin polynomial just takes the first n terms of the polynomial.

Example 19.5
What is the n^{th} Maclaurin polynomial for $f(x) = e^x$?

Solution. $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ since $f^{(k)}(x) = e^x$. ☐

Remark 19.6. If we take the limit as $n \to \infty$, we obtain the alternate expression for the exponential function: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

What happens when $n \to \infty$?
Well $P_n = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$. Let's come up with a better name for this.

Definition 19.7. $T_f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$ is the **Taylor series** of f centered on x_0.

Then we know for sure that $T_f(x_0) = f(x_0)$. What if $T_f(x) = f(x)$ for all x in the domain, just like the polynomials we have previously explored? Let's give these functions a name to refer to them more easily and formalize this.

Definition 19.8. f is **analytic** at x_0 if $T_f(x) = f(x)$ for all x in an open interval containing x_0.

Remark 19.9. Arithmetic operations, power, exponential, logarithmic, and trigonometric functions are all analytic.

Example 19.10 (Taylor Series of $\ln x$ near 1)

$\frac{d}{dx}\ln(x) = \frac{1}{x}$
$\ln''(x) = -x^{-2}$
$\ln^{(3)}(x) = 2x^{-3}$
$\ln^{(4)}(x) = -3 \cdot 2x^{-4}$
$\vdots$
$\ln^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}$.

$$\begin{aligned} T_{\ln}(x) &= 0 + 1(x-1) + \frac{1}{2}(-1)(x-1)^2 + \dots \\ &= \sum_{k=1}^{\infty}(-1)^{k_1}\frac{(k-1)!}{k!}(x-1)^k \\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k \end{aligned}$$

i.e.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

§20 October 12, 2018

Corollary 20.1 (The Alternating Harmonic Series)

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Example 20.2
Solve for the Taylor series of arctan at 0.

Solution. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$
$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots$, which converges for $|x| \leq 1$. So
$\arctan x + C = \int (1 - x^2 + x^4 - x^6 + \dots)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$, so
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ □

Corollary 20.3 (Alternate Form for π)
$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Example 20.4 (Taylor Series of sin and cos at 0)

Find $T_{\sin}(x)$ and $T_{\cos}(x)$

Solution.

$$\begin{aligned} & \sin(0) = && 0 \\ 1 = \; & \sin' 0 = && \cos 0 \\ 0 = \; & \sin'' 0 = && \cos' 0 \\ -1 = \; & \sin''' 0 = && \cos'' 0 \\ & \vdots & & \end{aligned}$$

$$\begin{aligned} T_{\sin}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} T_{\cos}(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \end{aligned}$$

□

Theorem 20.5 (Mean Value Theorem)

Let f be a differentiable function on $[a, b]$. Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

Theorem 20.6 (Taylor's Theorem with Remainder)

Let f be a smooth function with Taylor polynomials $P_0, P_1, P_2, \dots$ at x_0. Let $R_n(x) = f(x) - P_n(x)$. Let I be an interval containing x_0. Then for every $x \in I$, there exists c between x_0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x - x_0)^{n+1}$$

Sketch of Proof. Repeatedly apply the Mean Value Theorem. □

Corollary 20.7

If $\left| f^{(n+1)}(c) \right| \leq M$ for all c between x and x_0, then $|R_n(x)| \leq \frac{M}{(n+1)!} \cdot |x - x_0|^{n+1}$.

Example 20.8

Approximate $\sqrt{\frac{3}{2}}$ using the 3^{rd} Taylor polynomial for $\sqrt{x}$ centered at 1. Find an upper bound for the error.

Solution. The third Taylor polynomial for $\sqrt{x}$ centered at 1 is given by

$$P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3,$$

so $P_3(\frac{3}{2}) = \frac{157}{128} \approx 1.2266$ and the error is bounded by $\left|\frac{d^4}{dx^4}\sqrt{x}\right| = \left|-\frac{15}{16}x^{-\frac{7}{2}}\right| \leq \frac{15}{16}$ for $1 \leq \frac{3}{2}$, so

$$\left|R_3(\frac{3}{2})\right| \leq \frac{\frac{15}{16}}{4!}(\frac{3}{2}-1)^4 \approx 0.0025$$

□

§21 October 15, 2018

§21.1 Convergence of Power Series

Definition 21.1. A **power series** centered at $x = a$ is a series of the form

$$P(x) = \sum_{n=0}^{\infty} c_n \cdot (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where $c_0, c_1, c_2, \dots$ are real numbers.

Remark 21.2. A Taylor series centered at $x = a$ is a power series.

When does $P(x)$ converge?

Theorem 21.3 (Convergence of the Power Series)
Every power series $P(x)$ centered at $x = a$ falls into of three cases:

(i) $P(x)$ converges for $x = a$ and diverges for all $x \neq a$.

(ii) $P(x)$ converges for all real x.

(iii) There exists $R > 0$ such that $P(x)$ converges absolutely for all x satisfying $|x-a| < R$ and diverges for all x satisfying $|x-a| > R$.

Definition 21.4. The **interval of convergence** of $P(x)$ is $\{x \in \mathbb{R} \mid P(x) \text{ converges at } x\}$.

The **radius of convergence** is

- 0 in case (i)
- ∞ in case (ii)
- R in case (iii).

Example 21.5

Find the interval of convergence for

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=0}^{\infty} 2^{2^n} x^n$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(x-1)^n}{n}$.

Solution.

1. By the ratio test, $\rho = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$ for all x, so $P(x)$ converges for all real $x \in (-\infty, \infty)$.
2. By the ratio test, $\rho = \lim_{n\to\infty} \left|2^{2^n} x\right|$ which is 0 for $x = 0$ and ∞ otherwise. So $x \in \{0\}$.
3. $\rho = \lim_{n\to\infty} \left|\frac{(x-2)(n+1)}{3(n+2)}\right| = \left|\frac{x-2}{3}\right|$. For $\rho < 1$, we need $x \in (-1, 5)$.
4. $\rho = \lim_{n\to\infty} \left|-\frac{n(x-1)}{n+1}\right| = |x-1| < 1$ for $x \in (0, 2)$. □

§22 October 17, 2018

§22.1 Properties and Applications of Power Series

Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges to $f(x)$ on $I = (a-R, a+R)$ or $\mathbb{R}$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ converges to $g(x)$ on $J = (b-S, b+S)$ or $\mathbb{R}$. Then

(i) $\sum_{n=0}^{\infty} (c_n \pm d_n)(x-a)^n$ converges to $f(x) \pm g(x)$ on $I \cap J$, the intersection of I and J.

(ii) $\sum_{n=0}^{\infty} e_n (x-a)^n$, where $e_n = \sum_{k=0}^{n} c_k d_{n-k} = c_0 d_n + c_1 d_{n-1} + \ldots + c_n d_0$, converges to $f(x)g(x)$ for all $x \in I \cap J$.

(iii) $\sum_{n=0}^{\infty} c_n \left(\sum_{m=0}^{\infty} d_m (x-a)^m - a \right)^n$ converges to $f(g(x))$ for all $x \in J$ such that $g(x) \in I$.

(iv) $f(x)$ is differentiable on I and $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(v) $f(x)$ is integrable on I and $\int_a^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$

Example 22.1

What is a power series expression for $\dfrac{3}{(x+2)(x-1)}$?

Solution. Strategy: we can either take the product of two power series, or decompose the fraction into partial fractions and take the sum of the power series.

(i) $\dfrac{3}{(x+2)(x-1)} = \dfrac{3}{x+2} \cdot \dfrac{1}{x-1}$

$\dfrac{\frac{3}{2}}{1-(-\frac{x}{2})} \cdot \dfrac{-1}{1-x} = \left(\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n\right) \cdot \left(\sum_{n=0}^{\infty} -x^n\right)$

By the ratio test, the radius of convergence for the

(ii) $\dfrac{3}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$ and we find that $A = -1, B = 1$.

□

§23 October 19, 2018

§23.1 Applications of Taylor Series

Example 23.1

Evaluate $\left(\dfrac{d^{100}}{dx^{100}} \cos x^2\right)(0)$.

Solution. $\cos u = 1 - \dfrac{u^2}{2} + \dfrac{u^4}{4!} - \ldots$, so $\cos x^2 = 1 - \dfrac{x^4}{2} + \dfrac{x^8}{4!} - \ldots + (-1)^{25}\dfrac{x^{100}}{50!} + \ldots$ so the 100^{th} derivative evaluated at 0 is $(-1)^{25} \cdot \dfrac{1}{50!} \cdot f^{(100)}(0) = -\dfrac{100!}{50!}$, because all other terms go to 0. □

Example 23.2

Evaluate $\lim\limits_{x\to 0} \dfrac{\ln(1+x^{12})}{\sin^2 x^3 - x^6}$.

Solution. $\ln(1+x^{12}) = x^{12} - \dfrac{x^{24}}{2} + x^{36} \ldots$ for $-1 < x < 1$ and $\sin^2 u = \left(u - \dfrac{u^3}{6} + u^5 \ldots\right)^2 = u^2 - \dfrac{1}{3}u^4 + u^6 \ldots$, so $\sin^2 x^3 = x^6 - \dfrac{x^{12}}{3} + x^{18} \ldots$

$$\lim_{x\to 0} \frac{\ln(1+x^{12})}{\sin^2 x^3 - x^6} = \frac{x^{12} - \frac{x^{24}}{2} + \ldots}{-\frac{x^{12}}{3} + \ldots} = -3.$$

□

Proposition 23.3

If $f(x) = \sum_{n=K}^{\infty} c_n(x-a)^n$, $c_K \neq 0$ and $g(x) = \sum_{n=L}^{\infty} d_n(x-a)^n$, $d_L \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } K > L \\ \frac{c_k}{d_k} & \text{if } K = L \\ \text{undefined} & \text{if } K < L \end{cases}$$

Example 23.4 (Binomial Series)

$$\begin{aligned} f(x) &= (1+x)^r & r &\neq 0 \\ f'(x) &= r(1+x)^{r-1} & f(0) &= 1 \\ f''(x) &= r(r-1)(1+x)^{r-2} & f'(0) &= r \\ &\vdots \\ f^{(n)}(x) &= r(r-1)(r-2)\dots(r-n+1)(1+x)^{r-n} f^{(n)}(0) = r(r-1)(\dots(r-n+1) \end{aligned}$$

So $T_f(x) = 1 + rx + \frac{r(r-1)}{2}x^2 + \dots = \sum_{n=0}^{\infty}(rn)x^n$

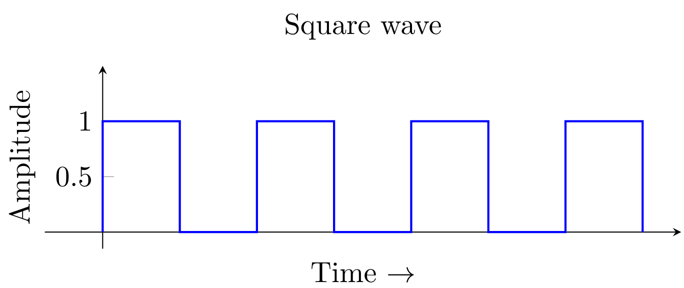
§24 October 24, 2018

§24.1 Fourier Polynomials

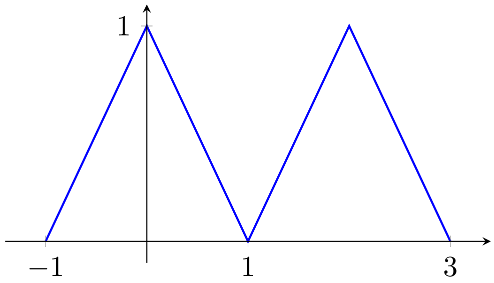
Taylor polynomials can approximate functions very well locally, but they often do pretty bad globally. In particular, Taylor polynomials cannot approximate periodic functions well globally.

Fourier polynomials, on the other hand, can approximate a function less well locally, but does a better job of approximating globally.

For example, the square wave function below is defined by $f(x) = \begin{cases} 0 & (2k-1)\pi \leq x \leq 2k\pi \\ 1 & 2k\pi \leq x \leq (2k+1)\pi \end{cases}$



The triangular wave function below is also a periodic function.



Main Idea: If f is a function with period 2π, we can approximate f by

$$\begin{aligned} f(x) \approx F_n(x) = a_0 + a_1 \cos x &\quad + b_1 \sin x \\ + a_2 \cos 2x &\quad + b_2 \sin 2x \\ + &\quad \vdots \\ + a_n \cos nx &+ b_n \sin nx. \end{aligned}$$

Finding Coefficients

$a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ — this is just the average value of f on $[-\pi, \pi]$

$a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin kx\,dx$

Example 24.1
Find the Fourier coefficients of the square wave.

Solution. $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$, $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} 1 \cdot \cos x\,dx = 0$, and $b_1 = \dfrac{1}{\pi}\displaystyle\int_0 o^{\pi} \sin x\,dx = \dfrac{2}{\pi}$. □

So for the square wave function, we have the Fourier polynomials:

$F_1 = \dfrac{1}{2} + \dfrac{2}{\pi}\sin x$

$F_2 = \dfrac{1}{2} + \dfrac{2}{\pi}\sin x$

$F_3 = \dfrac{1}{2} + \dfrac{2}{\pi}\sin x + \dfrac{2}{3\pi}\sin 3x$

Definition 24.2. The **Fourier series** for f is given by $a_0 + \displaystyle\sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} F_n$

Definition 24.3. The k^{th} **harmonic** is $a_k \cos kx + b_k \sin kx$.

Example 24.4 (Pulse-Train periodic Function)
Find the Fourier series of $f(x) = \begin{cases} 1 & 0 = x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} \le x \le 2\pi. \end{cases}$

Solution. $F_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x)\,dx = \dfrac{1}{2\pi}\cdot\dfrac{\pi}{2} = \dfrac{1}{4}$
$F_1 = \dfrac{1}{4} + \dfrac{1}{\pi}\cos x + \dfrac{1}{\pi}\sin x.$ □

§25 October 26, 2018

More about Fourier Series

Suppose f has period P. Consider f on $[x_0, x_0+P]$. Then the Fourier series of f is

$$a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2\pi}{P}kx\right) + b_k\sin\left(\frac{2\pi}{P}kx\right)\right)$$

where

$$a_0 = \frac{1}{p}\int_{x_0}^{x_0+P} f(x)\,dx$$

and for $k \geq 1$,

$$a_k = \frac{2}{P}\int_{x_0}^{x_0+P} f(x)\cos\left(\frac{2\pi}{P}kx\right)\,dx \qquad b_k = \frac{2}{P}\int_{x_0}^{x_0+P} f(x)\sin\left(\frac{2\pi}{P}kx\right)\,dx.$$

Example 25.1
Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x) = x - \lfloor x\rfloor$ for all $x\in\mathbb{R}$. Find the Fourier series of f.

Solution. $a_0 = \displaystyle\int_0^1 (x - \lfloor x\rfloor)\,dx = \int_0^1 x\,dx = \frac{1}{2}$
$a_k = 2\displaystyle\int_0^1 x\cos(2\pi kx)\,dx = 2\left(\frac{x\sin 2\pi kx}{2\pi k}\bigg|_0^1 - \int_0^1 \frac{\sin 2\pi kx}{2\pi k}\,dx\right) = 0 - 2\left(\frac{-\cos 2\pi kx}{(2\pi k)^2}\right)_0^1 =$ 0.[11]
$b_k = 2\displaystyle\int_0^1 x\sin(2\pi kx)\,dx = 2\left(-x\frac{\cos 2\pi kx}{2\pi k}\bigg|_0^1 + \int_0^1 \frac{\cos 2\pi kx}{2\pi k}\,dx\right)$
$= 2\left(-x\dfrac{\cos 2\pi kx}{2\pi k} + \dfrac{\sin 2\pi kx}{(2\pi k)^2}\,dx\right)_0^1 = -\dfrac{2}{2\pi k} = -\dfrac{1}{k\pi}.$

Thus, the Fourier series for the f is $\dfrac{1}{2} - \displaystyle\sum_{k=1}^{\infty}\frac{1}{k\pi}\sin 2\pi kx$ for $x\in(0,1)$. □

Theorem 25.2
Suppose $\displaystyle\int_{x_0}^{x_0+P} |f(x)|^2\,dx$ converges. Then the Fourier series of f converges to f "almost everywhere".

Remark 25.3. For engineering applications, it is safe to assume convergence

[11]This isn't surprising, since if we shift the graph by $\frac{1}{2}$, the sawtooth wave is just an odd function.

§26 October 29, 2018

§26.1 Energy of Periodic Signal

$$f(x) = a_0 + \sum_{k=1}^{\infty} \underbrace{(a_k \cos(kx) + b_k \sin(kx))}_{k^{th} \text{ harmonic}}$$

Definition 26.1. The **amplitude** of the k^{th} harmonic is $A_k = \sqrt{a_k^2 + b_k^2}$.

Definition 26.2. The **energy** of a periodic function with period 2π is

$$E(f)\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx.$$

Example 26.3 (Energy of the k^{th} harmonic)
$\frac{1}{\pi}\int_{-\pi}^{\pi} (a_k \cos(kx) + b_k \sin(kx))^2 \, dx = a_k^2 + b_k^2 = A_k^2$

Proof. $= \frac{1}{\pi}\int_{-\pi}^{\pi} (a_k^2 \cos^2(kx) + b_k^2 \sin^2(kx) + 2a_k b_k \sin(kx)\cos(kx))\, dx$
Without doing any calculations, we know that sin cos is odd, so taking the integral symmetric to 0 of this will just be 0.
$= \frac{1}{\pi}\left(a_k^2 \int_{-\pi}^{\pi} \cos^2(kx)\, dx + b_k^2 \int_{-\pi}^{\pi} \sin^2(kx)\, dx\right)$
$= \frac{1}{\pi}\left(a_k^2 \int_{-\pi}^{\pi} \frac{1+\cos(2kx)}{2}\, dx + b_k^2 \int_{-\pi}^{\pi} \frac{1-\cos(2kx)}{2}\, dx\right)$
$= a_k^2 + b_k^2 = A_k^2.$ □

$E(a_0) = \frac{1}{\pi} a_0^2 \, dx = 2a_0^2$, so define $A_0 = \sqrt{2} \cdot a_0$, so $E(a_0) = A_0^2$.

Theorem 26.4
$E(f) = A_0^2 + A_1^2 + A_2^2 + \ldots$

Definition 26.5. The **energy spectrum** is the graph of all (k, A_k^2).

Now we look at the square wave defined by:

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & -\pi \le x < 0 \end{cases}$$

The Fourier series of f is given by

$$\frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \cdot \sin((2n+1)x)$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

and its energy is given by

$$E(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx = \frac{1}{\pi}\int_0^{\pi} 1 \, dx = 1$$

§26.2 Parametric Equations

$x = x(t), y = y(t)$ for $t \in I$.
Then the parametric curve is given by $\{(x(t), y(t) \mid t \in I\}$

Example 26.6
$x(t) = \cos(t), y(t) = \sin(t)$ for $I = [0, 2\pi]$.

§27 October 31, 2018

§27.1 Parametric Equations/Curves

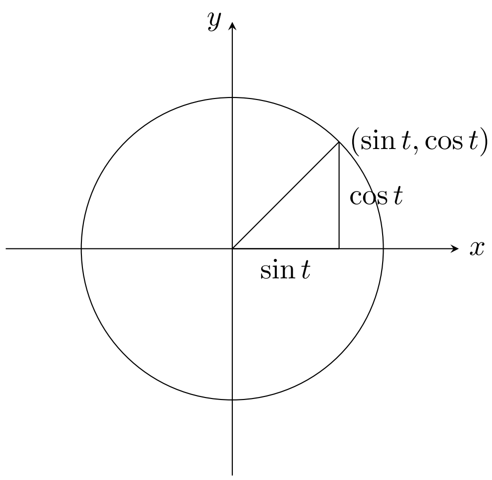
Definition 27.1. A **parametric equation** is a pair of functions

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

on an interval I.

Definition 27.2. The **graph** of the parametric function (x, y) is the set $\{(x(t), y(t)) \in \mathbb{R}^2 \mid t \in I\}$. This is also called a **parametric curve**.

Example 27.3
Let r be a real number and define $I = [0, 2\pi]$, $x(t) = r\cos t$, $y(t) = r\sin t$.
Then $\{(x(t), y(t)) \mid x^2 + y^2 = r^2\}$.



Example 27.4
Find the parametric curve defined by $I = [0, 2\pi]$ and $x(t) = 3\cos t$, $y(t) = -2\sin t$.

Solution. $x^2 = 9\cos^2 t = 9(1 - \sin^2 t)$ and $y^2 = 4\sin^2 t$
So $\frac{x^2}{9} + \frac{y^2}{4} = 1$. □

Remark 27.5. Notice that the negative sign in $y = -2\sin t$ did not change the graph from if $y = 2\sin t$. Even if this is the case, the parametrization of the graph is changed — the movement is now counterclockwise instead of clockwise. If we changed t to be $5t$, the graph is still the same, but now the movement is 5 times around the circle.

Example 27.6

Find the parametric curve defined by $x(t) = 3 - 4t\, y(t) = t + 7$.

Solution. $y - 7 = t \implies x = 3 - 4(y - 7) = -4y + 31$ which is just a line! □

Example 27.7

Find the parametric curve defined by $x(t) = t^2 - 1$, $y(t) = 3t + 2$.

Solution. $t = \frac{y-2}{3}$ so $x = \left(\frac{y-2}{3}\right)^2 - 1 = \left(\frac{y^2 - 4y + 4}{9}\right) - 1$ □

Example 27.8

Find the parametric curve defined by $x(t) = 3t^3 - 2\, y(t) = 5 + 2t^3$.

Example 27.9

Find the parametric curve defined by $x(t) = 3t^3 - 2\, y(t) = 5 + 2t^3$.

Remark 27.10. If we replace t^3 by t, the graph is still a line, but the movement is now slower because it is linear and not cubic. Note that if t^3 is replaced by t^2, then the graph becomes just a ray because squares can't be negative.

§28 November 2, 2018

§28.1 Calculus on Parametric Curves

Cycloid

Motion of center: $x_w(t) = at, \quad y_w(t) = a$.
Rotation of wheel: $x_{rot}(t) = a\cos(t + \frac{\pi}{2}) - -a\sin t \quad y_{rot}(t) = -a\sin(t + \frac{\pi}{2}) = -a\cos t$.
Parametrization of Cycloid:

$$x(t) = at - a\sin t$$
$$y(t) = a - a\cos t.$$

Definition 28.1. Let $x = x(t), y = y(t)$ be parametric equations. The derivative of $((x(t), y(t))$ is the vector $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$. This is also called the **velocity vector** and its magnitude $\sqrt{x'(t)^2 + y'(t)^2}$ is called the **speed**.

Thus, the velocity vector of the cycloid is given by $\begin{pmatrix} a - a\cos t \\ a\sin t \end{pmatrix}$.

Definition 28.2. The tangent line to $(x'(t), y(t))$ at $t = t_0$ is

$$T_{t_0}x(t) = x(t_0) + x'(t_0)(t - t_0)$$
$$T_{t_0}y(t) = y(t_0) + y'(t_0)(t - t_0).$$

Example 28.3

If $t_0 = \frac{3\pi}{2}$, then
$T_{\frac{3\pi}{2}}x(t) = \frac{3\pi}{2}a + a + (t - \frac{3\pi}{2})a = a + at.$
$T_{\frac{3\pi}{2}}y(t) = a - a(t - \frac{3\pi}{2}) = a - at + \frac{3\pi}{2}a.$

§29 November 9, 2018

$x(t) = r\cos t$
$y(t) = r\sin t$
$x'(t) = -r\sin t$
$y'(t) = r\cos t$
$x''(t) = -r\cos t = -x(t)$
$y''(t) = -r\sin t = -y(t)$

Remark 29.1. Velocity $\perp$ acceleration.

Remark 29.2. $\|(x'(t), y'(t))\| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$. In particular, this means constant speed.

Now we have $\Delta y(a_i) = y(a_{i+1}) - y(a_i)$ and $\Delta x(a_i) = x(a_{i+1}) - x(a_i)$. Then $L_i = \sqrt{\Delta y(a_i)^2 + \Delta x(a_i)^2}$.

Then $L = \int_{\text{curve}} \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Theorem 29.3

The arc length of a curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 29.4

What is the arc length of $y = 3x^3$ from $x = -1$ to 1.

Solution. The parametrization is given by $x = t, y = 3t^3$.
$\frac{1}{2}L = \int_0^1 \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{d}{dt}3t^3\right)^2} = \int_0^1 \sqrt{1 + (3t)^4}\, dt = \int_0^1 \sqrt{1 + \tan^2\theta}$ for $(3t)^2 = \tan\theta$. □

§30 November 12, 2018

Example 30.1

Find the arc length given by the parametric equation $x(t) = e^t - t, y(t) = 4e^{\frac{t}{2}}$ for $-8 \leq t \leq 3$.

Solution. The arc length is given by

$$\begin{aligned}
L &= \int_{-8}^{3} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_{-8}^{3} \sqrt{(e^t - 1)^2 + (2e^{\frac{t}{2}})^2}\, dt \\
&= \int_{-8}^{3} \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt \\
&= \int_{-8}^{3} \sqrt{(e^t + 1)^2}\, dt \\
&= \int_{-8}^{3} (e^t + 1)\, dt \\
&= (e^t + t)\Big|_{-8}^{3} \\
&= e^3 + 3 - e^{-8} + 8 \\
&\approx 31.
\end{aligned}$$

□

Let us approximate the area under a curve by summing up areas of rectangles under the curve.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} y(x(\overline{t_i})) \cdot (x(t_i) - x(t_{i-1})) \\
&= \sum_{i=1}^{n} y(x(\overline{t_i})) \cdot \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} y(x(\overline{t_i})) \cdot \frac{x(t_i) - x(t_{i-1})}{\Delta t} \Delta t
\end{aligned}$$

If we take the limit as $n \to \infty$, we obtain the following theorem:

Theorem 30.2

Let $x = x(t), y = y(t), a \leq t \leq b$ be a non-self-intersecting parametric curve with $x(t)$ a differentiable function. Then the area "under" the curve is given by

$$A = \int_a^b y(t)x'(t)\, dt.$$

Example 30.3

Find the area under $x(t) = t - \sin t, y(t) = 1 - \cos t$ for $0 \leq t \leq 2\pi$.

Solution.

$$\begin{aligned}
&\int_0^{2\pi} y(t)x'(t)\,dt \\
=&\int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\
=&\int_0^{2\pi} (1-2\cos t - \cos^2 t)\,dt \\
=&\int_0^{2\pi} (1-2\cos t - \frac{1+\cos 2t}{2}\,dt \\
=&\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\,dt \\
=&\left(\frac{3}{2} - 2\sin t + \frac{1}{4}\sin 2t\right)_0^{2\pi}
\end{aligned}$$

□